THE METHODOLOGY OF INDIAN MATHEMATICS

I. ALLEGED ABSENCE OF PROOFS IN INDIAN MATHEMATICS

Several books have been written on the history of Indian tradition in mathematics.[1] In addition, many books on history of mathematics devote a section, sometimes even a chapter, to the discussion of Indian mathematics. Many of the results and algorithms discovered by the Indian mathematicians have been studied in some detail. But, little attention has been paid to the methodology and foundations of Indian mathematics. There is hardly any discussion of the processes by which Indian mathematicians arrive at and justify their results and procedures. And, almost no attention is paid to the philosophical foundations of Indian mathematics, and the Indian understanding of the nature of mathematical objects, and validation of mathematical results and procedures.

Many of the scholarly works on history of mathematics assert that Indian Mathematics, whatever its achievements, does not have any sense of logical rigor. Indeed, a major historian of mathematics presented the following assessment of Indian mathematics over fifty years ago:

> The Hindus apparently were attracted by the arithmetical and computational aspects of mathematics rather than by the geometrical and rational features of the subject which had appealed so strongly to the Hellenistic mind. Their name for mathematics, *gaṇita*, meaning literally the 'science of calculation' well characterises this preference. They delighted more in the tricks that could be played with numbers than in the thoughts the mind could produce, so that neither Euclidean geometry nor Aristotelian logic made a strong impression upon them. The Pythagorean problem of the incommensurables, which was of intense interest to Greek geometers, was of little import to Hindu mathematicians, who treated rational and irrational quantities, curvilinear and rectilinear magnitudes indiscriminately. With respect to the development of algebra, this attitude occasioned perhaps and incremental advance, since by the Hindus the irrational roots of the quadratics were no longer disregarded as they had been by the Greeks, and since to the Hindus we owe also the immensely convenient concept of the absolute negative. These generalisations of the number system and the consequent freedom of arithmetic from geometrical representation were to be essential in the development of the concepts of calculus, but the Hindus could hardly have appreciated the theoretical significance of the change…

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[1] We may cite the following standard works: B. B. Datta and A. N. Singh, *History of Hindu Mathematics*, 2 Parts, Lahore 1935, 1938, Reprint, Delhi 1962; C. N. Srinivasa Iyengar, *History of Indian Mathematics*, Calcutta 1967; A. K. Bag, *Mathematics in Ancient and Medieval India*, Varanasi 1979; T. A. Saraswati Amma, *Geometry in Ancient and Medieval India*, Varanasi 1979; G. C. Joseph, *The Crest of the Peacock*: *The Non-European Roots of Mathematics,* 2nd Ed., Princeton 2000.

> The strong Greek distinction between the discreteness of number and the continuity of geometrical magnitude was not recognised, for it was superfluous to men who were not bothered by the paradoxes of Zeno or his dialectic. Questions concerning incommensurability, the infinitesimal, infinity, the process of exhaustion, and the other inquiries leading toward the conceptions and methods of calculus were neglected.[2]

Such views have found their way generally into more popular works on history of mathematics. For instance, we may cite the following as being typical of the kind of opinions commonly expressed about Indian mathematics:

> As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their name for mathematics was *gaṇita*, which means "the science of calculation". There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics.
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> It is fairly certain that the Hindus did not appreciate the significance of their own contributions. The few good ideas they had, such as separate symbols for the numbers from 1 to 9, the conversion to base 10, and negative numbers, were introduced casually with no realisation that they were valuable innovations. They were not sensitive to mathematical values. Along with the ideas they themselves advanced, they accepted and incorporated the crudest ideas of the Egyptians and Babylonians.[3]

The burden of scholarly opinion is such that even eminent mathematicians, many of whom have had fairly close interaction with contemporary Indian mathematics, have ended up subscribing to similar views, as may be seen from the following remarks of one of the towering figures of twentieth century mathematics:

> For the Indians, of course, the effectiveness of the *cakravāla* could be no more than an experimental fact, based on their treatment of great many specific cases, some of them of considerable complexity and

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[2] C.B.Boyer, *The History of Calculus and its Conceptual development*, New York 1949, p.61-62. As we shall see in the course of this article, Boyer's assessment – that the Indian mathematicians did not reach anywhere near the development of calculus or mathematical analysis, because they lacked the sophisticated methodology developed by the Greeks – seems to be thoroughly misconceived. In fact, in stark contrast to the development of mathematics in the Greco-European tradition, the methodology of Indian mathematical tradition seems to have ensured continued and significant progress in all branches of mathematics till barely two hundred years ago; it also led to major discoveries in calculus or mathematical analysis, without in anyway abandoning or even diluting its standards of logical rigour, so that these results, and the methods by which they were obtained, seem as much valid today as at the time of their discovery.

[3] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford 1972, p.190.

> involving (to their delight, no doubt) quite large numbers. As we shall see, Fermat was the first one to perceive the need for a general proof, and Lagrange was the first to publish one. Nevertheless, to have developed the *cakravāla* and to have applied it successfully to such difficult numerical cases as N=61, or N=67 had been no mean achievements.[4]

Modern scholarship seems to be unanimous in holding the view that Indian mathematics lacks any notion of proof. But even a cursory study of the source-works that are available in print would reveal that Indian mathematicians place much emphasis on providing what they refer to as *upapatti* (proof, demonstration) for every one of their results and procedures. Some of these *upapattis* were noted in the early European studies on Indian mathematics in the first half of the nineteenth Century. For instance, in 1817, H. T. Colebrooke notes the following in the preface to his widely circulated translation of portions of *Brahmasphuṭasiddhānta* of Brahmagupta and *Līlāvatī* and *Bījagaṇita* of Bhāskarācārya:

> On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by Bhāskara himself, towards the close of his algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.[5]

Another notice of the fact that detailed proofs are provided in the Indian texts on mathematics is due to Charles Whish who, in an article published in 1835, pointed out that infinite series for π and for trigonometric functions were derived in texts of Indian mathematics much before their 'discovery' in Europe. Whish concluded his paper with a sample proof from the Malayalam text *Yuktibhāṣā* of the theorem on the square of the diagonal of a right angled triangle and also promised that:

> A further account of the *Yuktibhāṣā*, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the

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4 André Weil, *Number Theory: An Approach through History from Hammurapi to Legendre,* Boston 1984, p.24. It is indeed ironical that Prof. Weil has credited Fermat, who is notorious for not presenting any proof for most of the claims he made, with the realisation that mathematical results need to be justified by proofs. While the rest of this article is purported to show that the Indian mathematicians presented logically rigorous proofs for most of the results and processes that they discovered, it must be admitted that the particular example that Prof. Weil is referring to, the effectiveness of the *cakravāla* algorithm (known to the Indian mathematicians at least from the time of Jayadeva, prior to the eleventh century) for solving quadratic indeterminate equations of the form $x^2 - Ny^2 = 1$, does not seem to have been demonstrated in the available source-works. In fact, the first proof of this result was given by Krishnaswamy Ayyangar barely seventy-five years ago (A.A. Krishnaswamy Ayyangar, 'New Light on Bhāskara's *Cakravāla* or Cyclic Method of solving Indeterminate Equations of the Second Degree in Two Variables', Jour Ind. Math. Soc. 18, 228-248, 1929-30). Krishnaswamy Ayyangar also showed that the *cakravāla* algorithm is different and more optimal than the Brouncker-Wallis-Euler-Lagrange algorithm for solving this so-called "Pell's Equation."

5 H T Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskara*, London 1817, p.xvii. Colebrooke also presents some of the *upapatti*s given by the commentators Gaṇeśa Daivajña and Kṛṣṇa Daivajña, as footnotes in his work.

> sines, cosines, and their demonstrations, will be given in a separate paper: I shall therefore conclude this, by submitting a simple and curious proof of the 47th proposition of Euclid [the so called Pythagoras theorem], extracted from the *Yuktibhāṣā*.[6]

It would indeed be interesting to find out how the currently prevalent view, that Indian mathematics lacks the notion of proof, obtained currency in the last 100-150 years.

II. Gaṇita: Indian Mathematics

Gaṇita, Indian mathematics, is defined as follows by Gaṇeśa Daivajña, in his famous commentary *Buddhivilāsinī* on Bhāskarācārya's *Līlāvatī*:

> *gaṇyate saṁkhyāyate tadgaṇitam. tatpratipādakatvena*
> *tatsaṁjñaṁ śāstram ucyate.* [7]

Gaṇita is calculation and numeration; and the science that forms the basis of this is also called *gaṇita*. *Gaṇita* is of two types: *vyaktagaṇita* and *avyaktagaṇita*. *Vyaktagaṇita*, also called *pāṭīgaṇita*, calculations on the board, is the branch of *gaṇita* that employs manifest quantities for performing calculations. *Avyaktagaṇita*, also called *Bījagaṇita*, takes recourse to the use of *avyakta* or unknown (indeterminate, un-manifest) quantities such as *yāvat-tāvat* (so much as), *kālaka* (black), *nīlaka* (blue) etc. The *avyakta* quantities are also called *varṇas* (colours) and are denoted by symbols *yā, kā, nī*, just as in modern algebra unknowns are denoted by symbols *x*, *y*, *z*, etc.

Gaṇita is generally taken to be a part of *jyotiḥśāstra*. Nṛsiṁha Daivajña, in his exposition, *vārttika,* on the commentary *Vāsanābhāṣya* by Bhāskarācārya on his own *Siddhāntaśiromaṇi*, says that the *gaṇita-skandha* of *jyotiḥśāstra* is composed of four types of *gaṇitas*: *vyakta-gaṇita, avyakta-gaṇita, graha-gaṇita* (mathematical astronomy which deals with calculation of planetary positions) and *gola-gaṇita* (spherical astronomy, which includes demonstrations of procedures of calculation using *gola*, the sphere, and *vedha*, observations). According to Gaṇeśa Daivajña the *prayojana* (purpose) of *gaṇita-śāstra* is 'the acquisition of knowledge concerning orbits, risings, settings, dimensions etc., of the planets and stars; and also the knowledge of *saṁhitā* (omens), *jātaka* (horoscopy), etc., which are indicators of the merits and demerits earned through the actions of former births'.[8]

It is not that the study of mathematics is entirely tied to astronomy. The ancient texts dealing with geometry, the *Śulva Sūtras* are part of *Kalpa*, a *Vedāṅga* different from

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6 C.M Whish, 'On the Hindu Quadrature of the Circle, and the infinite series of the proportion of the circumference to the diameter exhibited in the four Shastras, the Tantrasangraham, Yucti Bhasa, Carana Paddhati and Sadratnamala', Trans. Roy. As. Soc. (G.B.) 3, 509-523, 1835. However, Whish does not seem to have published any further paper on this subject.

7 *Buddhivilāsinī* of Gaṇeśa Daivajña (c.1545) on *Līlāvatī* of Bhāskarācārya II (c.1150), V.G. Apte (ed.), Vol I, Pune 1937, p. 5.

8 *Buddhivilāsinī*, cited above, p.7

Jyotiṣa, and deal with the construction of *yajña-vedīs*, alters. Much later, the Jaina mathematician Mahāvīrācārya (c.850) enumerates the uses of *gaṇita* in several contexts as follows:[9]

> In all transactions, which relates to worldly, Vedic or other similar religious affairs calculation is of use. In the science of love, in the science of wealth, in music and in drama, in the art of cooking, in medicine, in architecture, in prosody, in poetics and poetry, in logic and grammar and such other things, and in relation to all that constitutes the peculiar value of the arts, the science of calculation (*gaṇita*) is held in high esteem. In relation to the movement of the Sun and other heavenly bodies, in connection with eclipses and conjunctions of planets, and in connection with the *tripraśna* (direction, position and time) and the course of the Moon–indeed in all these it is utilised. The number, the diameter and the perimeter of islands, oceans and mountains; the extensive dimensions of the rows of habitations and halls belonging to the inhabitants of the world of light, of the world of the gods and of the dwellers in hell, and other miscellaneous measurements of all sort–all these are made out by the help of *gaṇita*. The configuration of living beings therein, the length of their lives, their eight attributes, and other similar things; their progress and other such things, their staying together, etc.–all these are dependent upon *gaṇita* (for their due comprehension). What is the good of saying much? Whatever there is in all the three worlds, which are possessed of moving and non-moving beings, cannot exist as apart from *gaṇita* (measurement and calculation).

The classification of *gaṇita* into *avyakta* and *vyakta* depends on whether indeterminate quantities like *yāvat-tāvat* etc. are employed in the various processes discussed. Thus *vyakta-gaṇita* subsumes not only arithmetic and geometry, but also topics included in 'algebra', such as solutions of equations, if no indeterminate quantities are introduced for finding the solutions. The celebrated text, *Līlāvatī* of Bhāskarācārya II (c.1150) deals with *vyakta-gaṇita* and is divided into the following sections: (1) *Paribhāṣā* (units and measures); (2) *Saṁkhyā-sthāna* (place-value system); (3) *Parikarmāṣṭaka* (eight operations of arithmetic, namely addition, subtraction, multiplication, division, square, square-root, cube, cube-root), (4) *Bhinna-parikarma* (operations with fractions); (5) *Śūnya-parikarma* (operations with zero), (6) *Prakīrṇa* (miscellaneous processes, including *trairāśika* (rule of three); (7) *Miśra-vyavahāra* (investigation of mixture, ascertaining composition as principal and interest joined and so forth); (8) *Śreḍhī- vyavahāra* (progressions and series); (9) *Kṣetra-vyavahāra* (plane geometry); (10) *Khāta- vyavahāra* (excavations and solids); (11) *Citi, Krakaca* and *Rāśi-vyavahāra* (calculation with stacks, saw, mounds of grain), (12) *Chāyā-vyavahāra* (gnomonics); (13) *Kuṭṭaka* (linear indeterminate equations); (14) *Aṅkapāśa* (combinatorics of digits).

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9 *Gaṇitasārasaṅgraha* of Mahāvīrācārya, 1.9-16, cited in B. B. Datta and A. N. Singh, Vol. I, 1935, cited earlier, p.5

The text *Bījagaṇita* of Bhāskarācārya deals with *avyakata-gaṇita* and is divided into the following sections: (1) *Dhanaṛṇa-ṣaḍvidha* (the six operations with positive and negative quantities, namely addition, subtraction, multiplication, division, square and square-root); (2) *Kha-ṣaḍvidha* (the six operations with zero); (3) *Avyakta-ṣaḍvidha* (the six operations with indeterminate quantities); (4) *Karaṇī-ṣaḍvidha* (the six operations with surds); (5) *Kuṭṭaka* (linear indeterminate equations); (6) *Varga-prakṛti* (quadratic indeterminate equation of the form $Nx^2+m=y^2$); (7) *Cakravāla* (cyclic process for the solution of above quadratic indeterminate equation); (8) *Ekavarṇa-samīkaraṇa* (simple equations with one unknown); (9) *Madhyamāharaṇa* (quadratic etc. equations); (10) *Anekavarṇa-samīkaraṇa* (simple equations with several unknowns); (11) *Madhyamāharaṇa-bheda* (varieties of quadratics); (12) *Bhāvita* (equations involving products). Here, the first seven sections, starting from *Dhanaṛṇa-ṣaḍvidha* to *Cakravāla* are said to be *bījopayogī* (adjuncts to algebraic analysis) and the last five sections deal with *bīja*, which is mainly of two types: *Ekavarṇa-samīkaraṇa* (equation with single unknown) and *Anekavarṇa-samīkaraṇa* (equation with several unknowns).

Gaṇeśa Daivajña raises the issue of the propriety of including discussion of *kuṭṭaka* (linear indeterminate equations) and *aṅkapāśa* (combinatorics), etc. in the work on *vyakta-gaṇita*, *Līlāvatī*, as they ought to be part of *Bījagaṇita*. He then goes on to explain that this is alright as an exposition of these subjects can be given without employing *avyakta-mārga*, i.e., procedures involving use of indeterminate quantities.

An interesting discussion of the relation between *vyakta* and *avyakta-gaṇita* is to be found in the commentary of Kṛṣṇa Daivajña on *Bījagaṇita* of Bhāskarācārya.[10] The statement of Bhāskara, '*vyaktam avyaktabījam*' can be interpreted in two ways: Firstly that *vyakta* is the basis of *avyakta* (*avyaktasya bījam*), because without the knowledge of *vyakta-gaṇita* (composed of addition, and other operations, the rule of three, etc.) one cannot even think of entering into a study of *avyakta-gaṇita*. It is also true that *vyakta* is that which is based on *avyakta* (*avyaktaṁ bījaṁ yasya*), because though the procedures of *vyakta-gaṇita* do not depend upon *avyakta* methods for being carried through (*svarūpa-nirvāha*), when it comes to justifying the *vyakta* methods by *upapattis* or demonstrations, *vyakta-gaṇita* is dependent on *avyakta-gaṇita*.

III *UPAPATTIS* IN INDIAN MATHEMATICS

The tradition of Upapattis in Mathematics and Astronomy

A major reason for our lack of comprehension, not merely of the Indian notion of proof, but also of the entire methodology of Indian mathematics, is the scant attention paid to the source-works so far. It is said that there are over one hundred thousand manuscripts on *Jyotiḥśāstra*, which includes, apart from works in *gaṇita-skandha* (mathematics and mathematical astronomy), also those in *saṁhitā-skandha* (omens) and *hora* (astrology).[11] Only a small fraction of these texts have been published. A

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[10] *Bījapallavam* commentary of Kṛṣṇa Daivajña on *Bījagaṇita* of Bhāskarācārya, T.V.Radhakrishna Sastri (ed.), Tanjore 1958, p.6-7.

[11] D. Pingree, *Jyotiḥśāstra: Astral and Mathematical Literature*, Wiesbaden 1981, p.118.

recent publication, lists about 285 published works in mathematics and mathematical astronomy. Of these, about 50 are from the period before 12th century AD, about 75 from 12th-15th centuries, and about 165 from 16th-19th centuries.[12]

Much of the methodological discussion is usually contained in the detailed commentaries; the original works rarely touch upon such issues. Modern scholarship has concentrated on translating and analysing the original works alone, without paying much heed to the commentaries. Traditionally the commentaries have played at least as great a role in the exposition of the subject as the original texts. Great mathematicians and astronomers, of the stature of Bhāskarācārya-I, Bhāskarācārya-II, Parameśvara, Nīlakaṇṭha Somasutvan, Gaṇeśa Daivajña, Munīśvara and Kamālakara, who wrote major original treatises of their own, also took great pains to write erudite commentaries on their own works and on works of earlier scholars. It is in these commentaries that one finds detailed *upapattis* of the results and procedures discussed in the original text, as also a discussion of the various methodological and philosophical issues. For instance, at the beginning of his commentary *Buddhivilāsinī,* Gaṇeśa Daivajña states:

> There is no purpose served in providing further explanations for the already lucid statements of Śrī Bhāskara. The knowledgeable mathematicians may therefore note the speciality of my intellect in the *upapattis*, which are after all the essence of the whole thing.[13]

Amongst the published works on Indian mathematics and astronomy, the earliest exposition of *upapattis* are to be found in the *bhāṣya* of Govindasvāmin (c 800) on *Mahābhāskarīya* of Bhāskarācārya-I, and the *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c 860) on *Brahmasphuṭasiddhānta* of Brahmagupta[14]. Then we find very detailed exposition of *upapattis* in the works of Bhāskarācārya-II (c.1150): his *Vivaraṇa* on *Śiṣyadhīvṛddhidātantra* of Lalla and his *Vāsanābhāṣya* on his own *Siddhāntaśiromaṇi*.[15] Apart from these, Bhāskarācārya provides an idea of what is an *upapatti* in his *Bījavāsanā* on his own *Bījagaṇita* in two places. In the chapter on *madhyamāharaṇa* (quadratic equations) he poses the following problem:

> Find the hypotenuse of a plane figure, in which the side and upright are equal to fifteen and twenty. And show the *upapattis* (demonstration) of the received procedure of computation.[16]

Bhāskarācārya provides two *upapattis* for the solution of this problem, the so-called Pythagoras theorem; and we shall consider them later. Again, towards the end of the

12 K. V. Sarma and B. V. Subbarayappa, *Indian Astronomy: A Source Book*, Bombay 1985.

13 *Buddhivilāsinī* of Gaṇeśa Daivajña, cited earlier, p.3

14 The *Āryabhaṭīyabhāṣya* of Bhāskara I (c.629) does occasionally indicate derivation of some of the mathematical procedures, though his commentary does not purport to present *upapattis* for the rules and procedures given in *Āryabhaṭīya*.

15 Ignoring all these classical works on *upapattis*, one scholar has recently claimed that the tradition of *upapatti* in India "dates from the 16th and 17th centuries" (J.Bronkhorst, 'Pāṇini and Euclid', Jour. Ind. Phil. 29, 43-80, 2001).

16 *Bījagaṇita* of Bhāskarācārya, Muralidhara Jha (ed.), Varanasi 1927, p. 69.

Bījagaṇita in the chapter on *bhāvita* (equations involving products), while considering integral solutions of equations of the form $ax + by = cxy$, Bhāskarācārya explains the nature of *upapatti* with the help of an example:

> The *upapatti* (demonstration) follows. It is twofold in each case: One geometrical and the other algebraic. The geometric demonstration is here presented…The algebraic demonstration is next set forth... This procedure has been earlier presented in a concise form by ancient teachers. The algebraic demonstrations are for those who do not comprehend the geometric one. Mathematicians have said that algebra is computation joined with demonstration; otherwise there would be no difference between arithmetic and algebra. Therefore this explanation of the principle of resolution has been shown in two ways.[17]

Clearly the tradition of exposition of *upapattis* is much older and Bhāskarācārya and later mathematicians and astronomers are merely following the traditional practice of providing detailed *upapattis* in their commentaries to earlier, or their own, works.

In Appendix I we give a list of important commentaries, available in print, which present detailed *upapattis*. It is unfortunate that none of the published source-works that we have mentioned above has so far been translated into any of the Indian languages, or into English; nor have they been studied in depth with a view to analyse the nature of mathematical arguments employed in the *upapattis* or to comprehend the methodological and philosophical foundations of Indian mathematics and astronomy.[18] In this article we present some examples of the kinds of *upapattis*

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[17] *Bījagaṇita*, cited above, p.125-127

[18] We may, however, mention the following works of C. T. Rajagopal and his collaborators, which provide an idea of the kind of *upapatti*s presented in the Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c 1530) for various results in geometry, trigonometry and those concerning infinite series for π and the trigonometric functions: K. Mukunda Marar, 'Proof of Gregory's Series', Teacher's Magazine 15, 28-34, 1940; K. Mukunda Marar and C. T. Rajagopal, 'On the Hindu Quadrature of the Circle', J.B.B.R.A.S. 20, 65-82, 1944; K. Mukunda Marar and C.T.Rajagopal, 'Gregory's Series in the Mathematical Literature of Kerala', Math. Student 13, 92-98, 1945; A. Venkataraman, 'Some Interesting proofs from *Yuktibhāṣā*', Math Student 16, 1-7, 1948; C. T. Rajagopal, 'A Neglected Chapter of Hindu Mathematics', Scr. Math. 15, 201-209, 1949; C. T. Rajagopal and A. Venkataraman, 'The Sine and Cosine Power Series in Hindu Mathematics', J.R.A.S.B. 15, 1-13, 1949; C. T. Rajagopal and T. V. V. Aiyar, 'On the Hindu Proof of Gregory's Series', Scr. Math. 17, 65-74, 1951; C.T.Rajagopal and T.V.V.Aiyar, 'A Hindu Approximation to Pi', Scr. Math. 18, 25-30, 1952; C.T.Rajagopal and M.S.Rangachari, 'On an Untapped Source of Medieval Keralese Mathematics', Arch. for Hist. of Ex. Sc. 18, 89-101, 1978; C. T. Rajagopal and M. S. Rangachari, 'On Medieval Kerala Mathematics', Arch. for Hist. of Ex. Sc. 35(2), 91-99, 1986. Following the work of Rajagopal and his collaborators, there are some recent studies which discuss some of the proofs in *Yuktibhāṣā*. We may cite the following: T. Hayashi, T.Kusuba and M.Yano, 'The Correction of the Mādhava Series for the Circumference of a Circle', Centauras, 33, 149-174, 1990; Ranjan Roy, 'The Discovery of the Series formula for π by Leibniz, Gregory and Nīlakaṇṭha', Math. Mag. 63, 291-306, 1990; V.J.Katz, 'Ideas of Calculus in Islam and India' Math. Mag. 68, 163-174, 1995; C.K.Raju, 'Computers, Mathematics Education, and the Alternative Epistemology of the Calculus in the *Yuktibhāṣā*', Phil. East and West 51, 325-362, 2001; D.F.Almeida, J.K.John and A.Zadorozhnyy, 'Keralese Mathematics: Its Possible Transmission to Europe and the Consequential Educational Implications', J. Nat. Geo. 20, 77-104, 2001; D.Bressoud, 'Was Calculus Invented in India?', College Math. J. 33, 2-13, 2002; J.K.John, 'Deriavation of the *Saṁskāras* applied to the Mādhava Series in *Yuktibhāṣā'*, in M.S.Sriram, K.Ramasubramanian and M.D.Srinivas (eds.), *500 Years of Tantrasaṅgraha*: *A Landmark in the*

provided in Indian mathematics, from the commentaries of Gaṇeśa Daivajña (c.1545) and Kṛṣṇa Daivajña (c.1600) on the texts *Līlāvatī* and *Bījagaṇita* respectively, of Bhāskarācārya -II (c.1150), and from the celebrated Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530). We shall also briefly discuss the philosophical foundations of Indian mathematics and its relation to other Indian sciences.

Mathematical results should be supported by Upapattis

Before discussing some of the *upapattis* presented in Indian mathematical tradition, it is perhaps necessary to put to rest the widely prevalent myth that the Indian mathematicians did not pay any attention to, and perhaps did not even recognise the need for justifying the mathematical results and procedures that they employed. The large corpus of *upapattis*, even amongst the small sample of source-works published so far, should convince anyone that there is no substance to this myth. Still, we may cite the following passage from Kṛṣṇa Daivajña's commentary *Bījapallavam* on *Bījagaṇita* of Bhāskarācārya, which clearly brings out the basic understanding of Indian mathematical tradition that citing any number of instances (even an infinite number of them) where a particular result seems to hold, does not amount to establishing that as a valid result in mathematics; only when the result is supported by a *upapatti* or a demonstration, can the result be accepted as valid:

> How can we state without proof (*upapatti*) that twice the product of two quantities when added or subtracted from the sum of their squares is equal to the square of the sum or difference of those quantities? That it is seen to be so in a few instances is indeed of no consequence. Otherwise, even the statement that four times the product of two quantities is equal to the square of their sum, would have to be accepted as valid. For, that is also seen to be true in some cases. For instance take the numbers 2, 2. Their product is 4, four times which will be 16, which is also the square of their sum 4. Or take the numbers 3, 3. Four times their product is 36, which is also the square of their sum 6. Or take the numbers 4, 4. Their product is 16, which when multiplied by four gives 64, which is also the square of their sum 8. Hence, the fact that a result is seen to be true in some cases is of no consequence, as it is possible that one would come across contrary instances also. Hence it is necessary that one would have to provide a proof (*yukti*) for the rule that twice the product of two quantities when added or subtracted from the sum of their squares results in the square of the sum or difference of those quantities. We shall provide the proof (*upapatti*) in the end of the section on *ekavarṇa-madhyamāharaṇa*.[19]

We shall now present a few *upapattis* as enunciated by Gaṇeśa Daivajña and Kṛṣṇa Daivajña in their commentaries on *Līlāvatī* and *Bījagaṇita* of Bhāskarācārya. These

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History of Astronomy, Shimla 2002, p. 169-182. An outline of the proofs given in *Yuktibhāṣā* can also be found in T. A. Saraswati Amma, 1979, cited earlier, and more exhaustively in S. Parameswaran, *The Golden Age of Indian Mathematics*, Kochi 1998.

[19] *Bījapallavam*, cited earlier, p.54.

upapattis are written in a technical Sanskrit, much like say the English of a text on Topology, and our translations below are somewhat rough renderings of the original.

The rule for calculating the square of a number

According to *Līlāvatī*:

> The multiplication of two like numbers together is the square. The square of the last digit is to be placed over it, and the rest of the digits doubled and multiplied by the last to be placed above them respectively; then omit the last digit, shift the number (by one place) and again perform the like operation…

Gaṇeśa's *upapatti* for the above rule is as follows:[20] On the left we explain how the procedure works by taking the example of $(125)^2 = 15,625$:

1	5	6	2	5	
				25	
			20		5^2
		4			2x2x5
		10			2^2
	4				2x1x5
1					2x1x2
					1^2
1	2	5			

By using the rule on multiplication, keeping in mind the place-values, and by using the mathematics of indeterminate quantities, let us take a number with three digits with *yā* at the 100[th] place, *kā* at the 10[th] place and *nī* at the unit place. The number is then [in the Indian notation with the plus sign understood] *yā 1 kā 1 nī 1*.

Using the rule for the multiplication of indeterminate quantities, the square [of the above number] will be *yā va 1 yā kā bhā 2 yā nī bhā 2 kā va 1 kā nī bhā 2 nī va 1* [using the Indian notation, where *va* after a symbol stands for *varga* or square and *bhā* after two symbols stands for *bhāvita* or product].

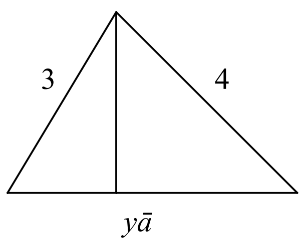
Here we see in the ultimate place, the square of the first digit *yā*; in second and third places there are *kā* and *nī* multiplied by twice the first *yā*. Hence the first part of the rule: "The square of the last digit…" Now, we see in the fourth place we have square of *kā*; in the fifth we have *nī* multiplied by twice *kā*; in the sixth we have square of *nī*. Hence it is said, "Then omitting the last digit move the number and again perform the like operation". Since we are finding the square by multiplying, we should add figures corresponding to the same place value, and hence we have to move the rest of the digits. Thus the rule is demonstrated.

[20] *Buddhivilāsinī*, cited above, p.19-20.

While Gaṇeśa provides such *avyaktarītya upapattis* or algebraic demonstrations for all procedures employed in arithmetic, Śaṅkara Vāriyar, in his commentary. *Kriyākramakarī*, presents *kṣetragata upapattis*, or geometrical demonstrations.

Square of the diagonal of a right-angled triangle; *the so-called Pythagoras Theorem*:

Gaṇeśa provides two *upapatti*s for calculating the square of the hypotenuse (*karṇa*) of a right-angled triangle.[21] These *upapatti*s are the same as the ones outlined by Bhāskarācārya-II in his *Bījavāsanā* on his own *Bījagaṇita*, and were referred to earlier. The first involves the *avyakta* method and proceeds as follows:[22]



$y\bar{a} = (9/y\bar{a}) + (16/y\bar{a})$

$y\bar{a}^2 = 25$

$y\bar{a} = 5$

Take the hypotenuse (*karṇa*) as the base and assume it to be *yā*. Let the *bhujā* and *koṭi* (the two sides) be 3 and 4 respectively. Take the hypotenuse as the base and draw the perpendicular to the hypotenuse from the opposite vertex as in the figure. [This divides the triangle into two triangles, which are similar to the original] Now by the rule of proportion (*anupāta*), if *yā* is the hypotenuse the *bhujā* is 3, then when this *bhujā* 3 is the hypotenuse, the *bhujā*, which is now the *ābādhā* (segment of the base) to the side of the original *bhujā* will be (9/*ya*). Again if *yā* is the hypotenuse, the *koṭi* is 4, then when this *koṭi* 4 is the hypotenuse, the *koṭi*, which is now the segment of base to the side of the (original) *koṭi* will be (16/*yā*). Adding the two segments (*ābādhās*) of *yā* the hypotenuse and equating the sum to (the hypotenuse) *yā,* cross-multiplying and taking the square-roots, we get *yā* = 5, the square root of the sum of the squares of *bhujā* and *koṭi*.

The other *upapatti* of Gaṇeśa is *kṣetragata* or geometrical, and proceeds as follows:[23]

Take four triangles identical to the given and taking the four hypotenuses to be the four sides, form the square as shown. Now, the interior square has for its side the difference of *bhujā* and

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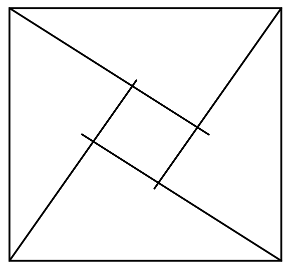
21 *Buddhivilāsinī*, cited earlier, p.128-129

22 Colebrooke remarks that this proof of the so-called Pythagoras theorem using similar triangles appeared in Europe for the first time in the work of Wallis in the seventeenth century (Colebrooke, cited earlier, p.xvi). The proof in Euclid's *Elements* is rather complicated and lengthy.

23 This method seems to be known to Bhāskarācārya-I (c. 629 AD) who gives a very similar diagram in his *Āryabhaṭīyabhāṣya*, K S Shukla (ed.), Delhi 1976, p.48. The Chinese mathematician Liu Hui (c 3rd Century AD) seems to have proposed similar geometrical proofs of the so-called Pythagoras Theorem. See for instance, D B Wagner, 'A Proof of the Pythagorean Theorem by Liu Hui', Hist. Math. 12, 71-3, 1985.



koṭi. The area of each triangle is half the product of *bhujā* and *koṭi* and four times this added to the area of the interior square is the area of the total figure. This is twice the product of *bhujā* and *koṭi* added to the square of their difference. This, by the earlier cited rule, is nothing but the sum of the squares of *bhujā* and *koṭi.* The square root of that is the side of the (big) square, which is nothing but the hypotenuse.

The rule of signs in Algebra

One of the important aspects of Indian mathematics is that in many *upapattis* the nature of the underlying mathematical objects plays an important role. We can for instance, refer to the *upapatti* given by Kṛṣṇa Daivajña for the well-known rule of signs in Algebra. While providing an *upapatti* for the rule, "the number to be subtracted if positive (*dhana*) is made negative (*ṛṇa*) and if negative is made positive", Kṛṣṇa Daivajña states:

> Negativity (*ṛṇatva*) here is of three types–spatial, temporal and that pertaining to objects. In each case, [negativity] is indeed the *vaiparītya* or the oppositeness… For instance, the other direction in a line is called the opposite direction (*viparīta dik*); just as west is the opposite of east… Further, between two stations if one way of traversing is considered positive then the other is negative. In the same way past and future time intervals will be mutually negative of each other…Similarly, when one possesses said objects they would be called his *dhana* (wealth). The opposite would be the case when another owns the same objects… Amongst these [different conceptions], we proceed to state the *upapatti* of the above rule, assuming positivity (*dhanatva*) for locations in the eastern direction and negativity (*ṛṇatva*) for locations in the west, as follows…[24]

Kṛṣṇa Daivajña goes on to explain how the distance between a pair of stations can be computed knowing that between each of these stations and some other station on the same line. Using this he demonstrates the above rule that "the number to be subtracted if positive is made negative…"

The kuṭṭaka process for the solution of linear indeterminate equations:

To understand the nature of *upapatti* in Indian mathematics one will have to analyse some of the lengthy demonstrations which are presented for the more complicated results and procedures. One will also have to analyse the sequence in which the results and the demonstrations are arranged to understand the method of exposition and logical sequence of arguments. For instance, we may refer to the demonstration given

[24] *Bījapallavam*, cited above, p.13.

by Kṛṣṇa Daivajña [25] of the celebrated *kuṭṭaka* procedure, which has been employed by Indian mathematicians at least since the time of Āryabhaṭa (c 499 AD), for solving first order indeterminate equations of the form

$$(ax + c)/b = y$$

where a, b, c are given integers and x, y are to be solved for integers. Since this *upapatti* is rather lengthy, it is presented separately as Appendix II. Here, we merely recount the essential steps. Kṛṣṇa Daivajña first shows that the solutions for x, y do not vary if we factor all three numbers a, b, c by the same common factor. He then shows that if a and b have a common factor then the above equation will not have a solution unless c is also divisible by the same common factor. Then follows the *upapatti* of the process of finding the greatest common factor of a and b by mutual division, the so-called Euclidean algorithm. He then provides an *upapatti* for the *kuṭṭaka* method of finding the solution by making a *vallī* (table) of the quotients obtained in the above mutual division, based on a detailed analysis of the various operations in reverse (*vyasta-vidhi*). Finally, he shows why the procedure differs depending upon whether there are odd or even number of coefficients generated in the above mutual division.

Nīlakaṇṭha's proof for the sum of an infinite geometric seires

In his *Āryabhaṭīyabhāṣya*, while deriving an interesting approximation for the arc of circle in terms of the *jyā* (sine) and the *śara* (versine), the celebrated Kerala Astronomer Nīlakaṇṭha Somasutvan presents a detailed demonstration of how to sum an infinite geometric series. Though it is quite elementary compared to the various other infinite series expansions derived in the works of the Kerala School, we shall present an outline of Nīlakaṇṭha's argument as it clearly shows how the notion of limit was well understood in the Indian mathematical tradition. Nīlakaṇṭha first states the general result [26]

$$(a/r) + (a/r)^2 + (a/r)^3 + ... = a/(r-1),$$

where the left hand side is an infinite geometric series with the successive terms being obtained by dividing by a *cheda* (common divisor), r assumed to be greater than 1. Nīlakaṇṭha notes that this result is best demonstrated by considering a particular case, say $r = 4$. Thus, what is to be demonstrated is that

$$(1/4) + (1/4)^2 + (1/4)^3 + ... = 1/3$$

Nīlakaṇṭha first obtains the sequence of results

$$1/3 = 1/4 + 1/(4.3)$$

[25] *Bijapallavam*, cited above, p.85-99.

[26] *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, *Gaṇitapāda*, K Sambasiva Sastri (ed.), Trivandrum 1931, p.142-143.

$$1/(4.3) = 1/(4.4) + 1/(4.4.3)$$

$$1/(4.4.3) = 1/(4.4.4) + 1/(4.4.4.3)$$

and so on, from which he derives the general result

$$1/3 - [1/4 + (1/4)^2 + ... + (1/4)^n] = (1/4^n)(1/3)$$

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between 1/3 and sum of powers of 1/ 4 (as given by the right hand side of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality

$$1/4 + (1/4)^2 + ... + (1/4)^n + ... = 1/3$$

Yuktibhāṣā proofs of infinite series for π and the trigonometric functions

One of the most celebrated works in Indian mathematics and astronomy, which is especially devoted to the exposition of *yukti* or proofs, is the Malayalam work *Yuktibhāṣā* (c.1530) of Jyeṣṭhadeva [27]. Jyeṣṭhadeva states that his work closely follows the renowned astronomical work *Tantrasaṅgraha* (c.1500) of Nīlakaṇṭha Somasutvan and is intended to give a detailed exposition of all the mathematics required thereof. The first half of *Yuktibhāṣā* deals with various mathematical topics in seven chapters and the second half deals with all aspects of mathematical astronomy in eight chapters. The mathematical part includes a detailed exposition of proofs for the infinite series and fast converging approximations for π and the trigonometric functions, which were discovered by Mādhava (c.1375). We present an outline of these extremely fascinating proofs in Appendix III.

IV. *UPAPATTI* AND "PROOF"

Mathematics as a search for infallible eternal truths

The notion of *upapatti* is significantly different from the notion of 'proof' as understood in the Greek and the modern Western tradition of mathematics. The *upapattis* of Indian mathematics are presented in a precise language and carefully display all the steps in the argument and the general principles that are employed. But while presenting the argument they make no reference whatsoever to any fixed set of axioms or link the argument to 'formal deductions' performed from such axioms. The *upapattis* of Indian mathematics are not formulated with reference to a formal axiomatic deductive system. Most of the mathematical discourse in the Greek as well

27 *Yuktibhāṣā* of Jyeṣṭhadeva, K. Chandrasekharan (ed.), Madras 1953. *Gaṇitādhyāya* alone was edited along with notes in Malayalam by Ramavarma Thampuran and A. R. Akhileswara Aiyer, Trichur 1947. The entire work is being edited, along with an ancient Sanskrit version, *Gaṇitayuktibhāṣā*, and English translation, by K.V.Sarma (in press).

as modern Western tradition is carried out with reference to some axiomatic deductive system. Of course, the actual proofs presented in mathematical literature are not presented in a formal system, but it is always assumed that the proof can be recast in accordance with the formal ideal.

The ideal of mathematics in the Greek and modern Western traditions is that of a formal axiomatic deductive system; it is believed that mathematics is and ought to be presented as a set of formal derivations from formally stated axioms. This ideal of mathematics is intimately linked with another philosophical presupposition–that mathematics constitutes a body of infallible absolute truths. Perhaps it is only the ideal of a formal axiomatic deductive system that could presumably measure up to this other ideal of mathematics being a body of infallible absolute truths. It is this quest for securing absolute certainty of mathematical knowledge, which has motivated most of the foundational and philosophical investigations into mathematics and shaped the course of mathematics in the Western tradition, from the Greeks to the contemporary times.

For instance, we may cite the popular mathematician philosopher of our times, Bertrand Russell, who declares, "I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere." In a similar vein, David Hilbert, one of the foremost mathematicians of our times declared, "The goal of my theory is to establish once and for all the certitude of mathematical methods."[28]

A recent book recounts how the continued Western quest for securing absolute certainty for mathematical knowledge originates from the classical Greek civilisation:

> The roots of the philosophy of mathematics, as of mathematics itself, are in classical Greece. For the Greeks, mathematics meant geometry, and the philosophy of mathematics in Plato and Aristotle is the philosophy of geometry. For Plato, the mission of philosophy was to discover true knowledge behind the veil of opinion and appearance, the change and illusion of the temporal world. In this task, mathematics had a central place, for mathematical knowledge was the outstanding example of knowledge independent of sense experience, knowledge of eternal and necessary truths.[29]

The Indian epistemological view

Indian epistemological position on the nature and validation of mathematical knowledge is very different from that in the Western tradition. This is brought out for instance by the Indian understanding of what an *upapatti* achieves. Gaṇeśa Daivajña declares in his preface to *Buddhivilāsinī* that:

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28 Both quotations cited in Reuben Hersh, *Some Proposals for Reviving the Philosophy of Mathematics*, Adv. Math. 31,31-50,1979.

29 Philips J. Davis and Reuben Hersh, *The Mathematical Experience*, Boston 1981, p.325

Vyaktevā'vyaktasaṁjñe yaduditamakhilaṁ nopapattiṁ vinā tan-nirbhrānto vā ṛte tāṁ sugaṇakasadasi prauḍhatāṁ naiti cāyam pratyakṣaṁ dṛśyate sā karatalakalitādarśavat suprasannā tasmādagryopapattiṁ nigaditumakhilam utsahe buddhivṛddhyai:[30]

Whatever is stated in the *vyakta or avyakta* branches of mathematics, without *upapatti*, will not be rendered *nir-bhrānta* (free from confusion); will not have any value in an assembly of mathematicians. The *upapatti* is directly perceivable like a mirror in hand. It is therefore, as also for the elevation of the intellect (*buddhi-vṛddhi*), that I proceed to enunciate *upapattis* in entirety.

Thus the purpose of *upapatti* is: (i) To remove confusion in the interpretation and understanding of mathematical results and procedures; and, (ii) To convince the community of mathematicians. These purposes are very different from what a 'proof' in Western tradition of mathematics is supposed to do, which is to establish the 'absolute truth' of a given proposition.

In the Indian tradition, mathematical knowledge is not taken to be different in any 'fundamental sense' from that in natural sciences. The valid means for acquiring knowledge in mathematics are the same as in other sciences: *Pratyakṣa* (perception), *Anumāna* (inference), *Śabda* or *Āgama* (authentic tradition). Gaṇeśa's statement above on the purpose of *upapatti* follows the earlier statement of Bhāskarācārya-II. In the beginning of the *golādhyāya* of *Siddhāntaśiromaṇi*, Bhāskarācārya says:

madhyādyaṁ dyusadāṁ yadatra gaṇitam tasyopapattiṁ vinā
prauḍhiṁ prauḍhasabhāsu naiti gaṇako niḥsamśayo na svayam
gole sā vimalā karāmalakavat pratyakṣato dṛśyate
tasmādasmyupapattibodhavidhaye golaprabandhodyataḥ[31]

Without the knowledge of *upapattis*, by merely mastering the *gaṇita* (calculational procedures) described here, from the *madhyamādhikara* (the first chapter of *Siddhāntaśiromaṇi*) onwards, of the (motion of the) heavenly bodies, a mathematician will not have any value in the scholarly assemblies; without the *upapattis* he himself will not be free of doubt (*niḥsamśaya*). Since *upapatti* is clearly perceivable in the (armillary) sphere like a berry in the hand, I therefore begin the *golādhyāya* (section on spherics) to explain the *upapattis*.

As the commentator Nṛsiṁha Daivajña explains, 'the *phala* (object) of *upapatti* is *pāṇḍitya* (scholarship) and also removal of doubts (for oneself) which would enable one to reject wrong interpretations made by others due to *bhrānti* (confusion) or otherwise.' [32]

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[30] *Buddhivilāsinī*, cited above, p.3

[31] *Siddhāntaśiromaṇi* of Bhāskarācārya with *Vāsanābhāṣya* and *Vāsanāvārttika* of Nṛsiṁha Daivajña, Muralidhara Chaturveda (ed.), Varanasi 1981, p.326

[32] *Siddhāntaśiromaṇi*, cited above, p.326

In his *Vāsanābhāṣya* on *Siddhāntaśiromaṇi*, Bhāskarācārya refers to the sources of valid knowledge (*pramāṇa*) in mathematical astronomy, and declares that

> *yadyevamucyate gaṇitaskandhe upapattimān āgama eva pramāṇam*[33]
>
> Whatever is discussed in mathematical astronomy, the *pramāṇa* is authentic tradition or established text supported by *upapatti*.

Upapatti here includes observation. Bhāskarācārya, for instance, says that the *upapatti* for the mean periods of planets involves observations over very long periods.

Upapatti thus serves to derive and clarify the given result or procedure and to convince the student. It is not intended to be an approximation to some ideal way of establishing the absolute truth of a mathematical result in a formal manner starting from a given set of self-evident axioms. *Upapattis* of Indian mathematics also depend on the context and purpose of enquiry, the result to be demonstrated, and the audience for whom the *upapatti* is meant.

An important feature that distinguishes the *upapattis* of Indian mathematicians is that they do not employ the method of proof by contradiction or *reductio ad absurdum*. Sometimes arguments, which are somewhat similar to the proof by contradiction, are employed to show the non-existence of an entity, as may be seen from the following *upapatti* given by Kṛṣṇa Daivajña to show that "a negative number has no square root":

> The square-root can be obtained only for a square. A negative number is not a square. Hence how can we consider its square-root? It might however be argued: 'Why will a negative number not be a square? Surely it is not a royal fiat'… Agreed. Let it be stated by you who claim that a negative number is a square as to whose square it is; surely not of a positive number, for the square of a positive number is always positive by the rule… not also of a negative number. Because then also the square will be positive by the rule… This being the case, we do not see any such number whose square becomes negative…[34]

Such arguments, known as *tarka* in Indian logic, are employed only to prove the non-existence of certain entities, but not for proving the existence of an entity, which existence is not demonstrable (at least in principle) by other direct means of verification. In rejecting the method of indirect proof as a valid means for establishing existence of an entity which existence cannot even in principle be established through any direct means of proof, the Indian mathematicians may be seen as adopting what is nowadays referred to as the 'constructivist' approach to the issue of mathematical existence. But the Indian philosophers, logicians, etc., do much more than merely disallow certain existence proofs. The general Indian philosophical position is one of eliminating from logical discourse all reference to such *aprasiddha* entities, whose

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[33] *Siddhāntaśiromaṇi*, cited above, p. 30

[34] *Bījapallavam*, cited earlier, p.19.

existence in not even in principle accessible to all means of verification.[35] This appears to be also the position adopted by the Indian mathematicians. It is for this reason that many an "existence theorem" (where all that is proved is that the non-existence of a hypothetical entity is incompatible with the accepted set of postulates) of Greek or modern Western mathematics would not be considered significant or even meaningful by Indian mathematicians.

A new epistemology for Mathematics

Mathematics today, rooted as it is in the modern Western tradition, suffers from serious limitations. Firstly, there is the problem of 'foundations' posed by the ideal view of mathematical knowledge as a set of infallible absolute truths. The efforts of mathematicians and philosophers of the West to secure for mathematics the status of indubitable knowledge has not succeeded; and there is a growing feeling that this goal may turn out to be a mirage.

Apart from the problems inherent in the goals set for mathematics, there are also other serious inadequacies in the Western epistemology and philosophy of mathematics. The ideal view of mathematics as a formal deductive system gives rise to serious distortions. Some scholars have argued that this view of mathematics has rendered philosophy of mathematics barren and incapable of providing any understanding of the actual history of mathematics, the logic of mathematical discovery and, in fact, the whole of creative mathematical activity. According one philosopher of mathematics:

> Under the present dominance of formalism, the school of mathematical philosophy which tends to identify mathematics with its formal axiomatic abstraction and the philosophy of mathematics with meta-mathematics, one is tempted to paraphrase Kant: The history of mathematics, lacking the guidance of philosophy, has become blind, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics has become empty… The history of mathematics and the logic of mathematical discovery, i.e., the phylogenesis and the ontogenesis of mathematical thought, cannot be developed without the criticism and ultimate rejection of formalism…[36]

There is also the inevitable chasm between the ideal notion of infallible mathematical proof and the actual proofs that one encounters in standard mathematical practice, as portrayed in a recent book:

> On the one side, we have real mathematics, with proofs, which are established by the 'consensus of the qualified'. A real proof is not checkable by a machine, or even by any mathematician not privy to the *gestalt*, the mode of thought of the particular field of mathematics in

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[35] For the approach adopted by Indian philosophers to *tarka* or the method of indirect proof, see the preceding article on 'Indian Approach to Logic'.

[36] I. Lakatos, *Proofs and Refutations*: *The Logic of Mathematical Discovery*, Cambridge 1976, p.1

> which the proof is located. Even to the 'qualified reader' there are normally differences of opinion as to whether a real proof (i.e., one that is actually spoken or written down) is complete or correct. These doubts are resolved by communication and explanation, never by transcribing the proof into first order predicate calculus. Once a proof is 'accepted', the results of the proof are regarded as true (with very high probability). It may take generations to detect an error in a proof… On the other side, to be distinguished from real mathematics, we have 'meta-mathematics'… It portrays a structure of proofs, which are indeed infallible 'in principle'… [The philosophers of mathematics seem to claim] that the problem of fallibility in real proofs… has been conclusively settled by the presence of a notion of infallible proof in meta-mathematics… One wonders how they would justify such a claim.[37]

Apart from the fact that the modern Western epistemology of mathematics fails to give an adequate account of the history of mathematics and standard mathematical practice, there is also the growing awareness that the ideal of mathematics as a formal deductive system has had serious consequences in the teaching of mathematics. The formal deductive format adopted in mathematics books and articles greatly hampers understanding and leaves the student with no clear idea of what is being talked about.

Notwithstanding all these critiques, it is not likely that, within the Western philosophical tradition, any radically different epistemology of mathematics will emerge and so the driving force for modern mathematics is likely to continue to be a search for absolute truths and modes of establishing them, in one form or the other. This could lead to 'progress' in mathematics, but it would be progress of a rather limited kind.

If there is a major lesson to be learnt from the historical development of mathematics, it is perhaps that the development of mathematics in the Greco-European tradition was seriously impeded by its adherence to the cannon of ideal mathematics as laid down by the Greeks. In fact, it is now clearly recognised that the development of mathematical analysis in the Western tradition became possible only when this ideal was given up during the heydays of the development of "infinitesimal calculus" during 16th and 18th centuries. As one historian of mathematics notes:

> It is somewhat paradoxical that this principal shortcoming of Greek mathematics stemmed directly from its principal virtue–the insistence on absolute logical rigour…Although the Greek bequest of deductive rigour is the distinguishing feature of modern mathematics, it is arguable that, had all the succeeding generations also refused to use real numbers and limits until they fully understood them, the calculus might never have been developed and mathematics might now be a dead and forgotten science.[38]

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[37] Philip J. Davis and Reuben Hersh, 1981, cited earlier, p.354-5

[38] C.H. Edwards, *History of Calculus*, New York 1979, p.79

It is of course true that the Greek ideal has gotten reinstated at the heart of mathematics during the last two centuries, but it seems that most of the foundational problems of mathematics can also be perhaps traced to the same development. In this context, study of alternative epistemologies such as that developed in the Indian tradition of mathematics, could prove to be of great significance for the future of mathematics.

V. THE NATURE OF MATHEMATICAL OBJECTS

Another important foundational issue in mathematics is that concerning the nature of mathematical objects. Here again the philosophical foundations of contemporary mathematics are unsatisfactory, with none of the major schools of thought, namely Platonism, Formalism or Intuitionism, offering a satisfactory account of the nature of mathematical objects (such as numbers) and their relation to other objects in the universe.

In the Indian tradition, the ontological status of mathematical objects such as numbers and their relation to other entities of the world was investigated in depth by the different schools of Indian philosophy. In the Western tradition, on the other hand, there was no significant discussion on the nature of numbers after the Greek times, till the work of Frege in the nineteenth century.[39] Recently, many scholars have explained that the *Nyāya* theory of numbers is indeed a highly sophisticated one, and may even prove in some respects superior to Frege's theory or the later developments, which have followed from it.[40]

In *Nyāya-vaiśeṣika* ontology, *saṁkhyā* or number-property is assigned to the category *guṇa* (usually translated as quality) which resides in *dravya* (translated as substance) by the relation of *samavāya* (translated as inherence). A *saṁkhyā* such as *dvitva* (two-ness or duality) is related to each of the objects of a pair by *samavāya*, and gives raise to the *jñāna* or cognition: '*ayaṁ dvitvavān*: This (one) is (a) locus of two-ness'. Apart from this, the number-property, *dvitva* (two-ness) is related to both the objects together by a relation called *paryāpti* (completion) and gives rise to the cognition *'imau dvau*: These are two'. So according to *Nyāya*, there are two ways in which number-properties such as one-ness or unity, two-ness or duality, three-ness, etc., are

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39 Ironically, Frege himself laid the blame for the unsatisfactory character of arithmetic that prevailed in his times and earlier, to the fact that its method and concepts originated in India: "After deserting for a time the old Euclidean standards of rigour, mathematics, is now returning to them, and even making efforts to go beyond them. In arithmetic, if only because many of its methods and concepts originated in India, it has been the tradition to reason less strictly than in geometry which was in the main developed by the Greeks."(G. Frege, *Foundations of Arithmetic* (original German edition, Bresslau 1884), Eng. tr. J. L. Austin, Oxford 1956, p.1e)

40 See, for instance, D. H. H. Ingalls, *Materials for the study of Navya-Nyāya Logic*, Harvard 1951; D. C. Guha, *Navya-Nyāya System of Logic*, Varanasi 1979; J. L. Shaw, 'Number: From *Nyāya* to Frege-Russel', Studia Logica, 41, 283-291, 1981; Roy W. Perret, 'A Note on the *Navya-Nyāya* Account of Number', Jour. Ind. Phil. 15, 227-234, 1985; B. K. Matilal, 'On the Theory of Number and *Paryāpti* in *Navya-Nyāya*', J.A.S.B, 28, 13-21, 1985; J.Ganeri, 'Numbers as Properties of Objects: Frege and the *Nyāya*', Stud. in Hum. and Soc. Sc. 3, 111-121, 1996.

related to a group of things numbered–firstly by the *samavāya* relation with each thing and secondly by the *paryāpti* relation with all the things together.

The *paryāpti* relation, connecting the number-property to the numbered things together, is taken by the Naiyāyikas to be a *svarūpa-saṁbandha*, where the two terms of the relation are identified ontologically. Thus, any number property such as two-ness is not unique; there are indeed several "two-nesses", there being a distinct two-ness associated (and identified) with every pair of objects. There are also the universals such as *dvitvatva* ("two-ness-ness"), which inhere in each particular two-ness.

The Naiyāyika theory of number and *paryāpti* was developed during 16^{th}-19^{th} centuries, in the context of developments in logic. The fact that the Naiyāyikas employ the relation of *paryāpti* by which number-property such as two-ness resides in both the numbered objects together and not in each one of them, has led various scholars to compare the *Nyāya* formulation with Frege's theory of numbers. According to Bertrand Russell's version of this theory, there is a unique number two, which is the set of all sets of two elements (or set of all pairs of objects). Thus the number two is a set of 'second-order', somewhat analogous to the universal "two-ness-ness" (*dvitvatva*) of the Naiyāyikas.

The Naiyāyika theory, however, differs from the modern Western formulations in that the Naiyāyikas employ the concept of property (*guṇa*), which has a clearly specified ontological status, and avoid notions such as 'sets' whose ontology is dubious. Any number-property such as two-ness associated with a pair of objects is ontologically identified with the pair, or both the objects together, and not with any 'set' (let alone the set of all sets) constituted by such a pair.

Apart from their theory of numbers, the general approach of the Indian logicians is what may be referred to as 'intentional', as opposed to the 'extensional' approach of most of Western logic and mathematics. Indian logicians have built a powerful system of logic, which is able to handle properties as they are (with both their intentions and extensions) and not by reducing them to classes (which are pure extensions, with the intentions being abstracted away); perhaps it can also help in clarifying the nature of mathematical objects and mathematical knowledge.

It is widely accepted that:

> Mathematics, as it exists today, is extensional rather than intentional. By this we mean that, when a propositional function enters into a mathematical theory, it is usually the extension of the function (i.e., the totality of entities or sets of entities that satisfy it) rather than its intention (i.e., its "context" or meaning) that really matters. This leaning towards extensionality is reflected in a preference for the language of classes or sets over the formally equivalent language of predicates with a single argument…[41]

[41] G. T. Kneebone, *Mathematical Logic and the Foundations of Mathematics*, London 1963, p.117

If the elementary propositions of the theory are of the form F(x) ("*x* has *F*", where *F* is the predicate with a single argument *x* which runs over a domain of 'individuals'), then it is but a matter of preference whether we use the language of predicates or of classes; each predicate corresponds to the class of all those individuals which satisfy the corresponding predicate. However the elementary propositions of Indian logic are of the form *xRy*, which relate any two 'entities' (not necessarily 'individuals') *x*, *y* by a relation *R*. The elementary proposition in Indian logic is always composed of a *viśeṣya* (qualificand *x*), *viśeṣaṇa* or *prakāra* (qualifier *y*) and a *saṁsarga* (relation *R*). Here *y* may also be considered as a *dharma* (property) residing in *x* by the relation *R*. Using these and other notions, Indian logicians developed a precise technical language, based on Sanskrit, which is unambiguous and makes transparent the logical structure of any (complex) proposition and which is used in some sense like the symbolic formal languages of modern mathematical logic.[42]

The dominant view, concerning the nature of mathematics today, is essentially that adopted by Bourbaki:

> Mathematics is understood by Bourbaki as a study of structures, or systematic of relations, each particular structure being characterised by a suitable set of axioms. In mathematics, as it exists at the present time, there are three great families of structures… namely algebraic structures, topological structures and ordinal structures. Any particular structure is to be thought of as inhering in a certain set...which functions as a domain of individuals for the corresponding theory.[43]

Bourbaki presents the whole of mathematics as an extension of the theory of sets. But if the study of abstract structures is indeed the goal of mathematics, there is no reason why this enterprise should necessarily be based on the theory of sets, unless one does not have the appropriate logical apparatus to handle philosophically better founded concepts such as properties, relations etc. The endeavour of the Indian logicians was to develop such a logical apparatus. This apparatus seems highly powerful and relevant for evolving a sounder theory of mathematical structures.

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[42] See for instance the previous article on 'Indian Approach to Logic'.

[43] G. T. Kneebone, cited earlier, p.326

APPENDIX I: LIST OF WORKS CONTAINING *UPAPATTIS*

The following are some of the important commentaries available in print, which present *upapattis* of results and procedures in mathematics and astronomy:

1. *Bhāṣya* of Bhāskara I (c.629) on *Āryabhaṭīya* of Āryabhaṭa (c.499), K.S.Shukla (ed.), New Delhi 1975.
2. *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629), T. S. Kuppanna Sastri (ed.), Madras 1957.
3. *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c.860) on *Brahmasphuṭasiddhānta* of Brahmagupta (c.628), Chs. I-III, XXI, Ramaswarup Sharma (ed.), New Delhi 1966; Ch XXI, Edited and Translated by Setsuro Ikeyama, Ind. Jour Hist. Sc. Vol. 38, 2003.
4. *Vivaraṇa* of Bhāskarācārya II (c.1150) on *Śiṣyadhīvṛddhidātantra* of Lalla (c.748), Chandrabhanu Pandey (ed.), Varanasi 1981.
5. *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Bījagaṇita*, Jivananda Vidyasagara (ed.), Calcutta 1878; Achyutananda Jha (ed.), Varanasi 1949, Rep. 1994.
6. *Mitākṣarā* or *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Siddhāntaśiromaṇi*, Bapudeva Sastrin (ed.) Varanasi 1866; Muralidhara Chaturveda (ed.), Varanasi 1981.
7. *Vāsanābhāṣya* of Āmarāja (c.1200) on *Khaṇḍakhādyaka* of Brahmagupta (c.665), Babuaji Misra (ed.), Calcutta 1925.
8. *Gaṇitabhūṣaṇa* of Makkībhaṭṭa (c.1377) on *Siddhāntaśekhara* of Śrīpati (c.1039), Chs. I – III, Babuaji Misra (ed.), Calcutta 1932
9. *Siddhāntadīpikā* of Parameśvara (c.1431) on the *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629), T. S. Kuppanna Sastri (ed.), Madras 1957.
10. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somasutvan (c.1501) on *Āryabhaṭīya* of Āryabhaṭa, K. Sambasiva Sastri (ed.), 3 Vols., Trivandrum 1931, 1932, 1957.
11. *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530); *Gaṇitādhyaya*, Ramavarma Thampuran and A.R. Akhileswara Aiyer (eds.), Trichur 1947; K. Chandrasekharan (ed.), Madras 1953; Edited, along with an ancient Sanskrit version *Gaṇitayuktibhāṣā* and English Translation, by K.V.Sarma (in Press).
12. *Yuktidīpikā* of Śaṅkara Vāriyar (c.1530) on *Tantrasaṅgraha* of Nīlakaṇṭha Somasutvan (c.1500), K.V. Sarma (ed.), Hoshiarpur 1977.
13. *Kriyākramakarī* of Śaṅkara Vāriyar (c.1535) on *Līlāvatī* of Bhāskarācārya II (c.1150), K.V. Sarma (ed.), Hoshiarpur 1975.
14. *Sūryaprakāśa* of Sūryadāsa (c.1538) on Bhāskarācārya's *Bījagaṇita* (c.1150), Chs. I – V, Edited and translated by Pushpa Kumari Jain, Vadodara 2001.
15. *Buddhivilāsinī* of Gaṇeśa Daivajña (c.1545) on *Līlāvatī* of Bhāskarācārya II (c.1150), V.G. Apte (ed.), 2 Vols, Pune 1937.
16. *Ṭīkā* of Mallāri (c.1550) on *Grahalāghava* of Gaṇeśa Daivajña (c.1520), Bhalachandra (ed.), Varanasi 1865; Kedaradatta Joshi (ed.), Varanasi 1981.
17. *Bījanavāṅkurā* or *Bījapallavam* of Kṛṣṇa Daivajña (c.1600) on *Bījagaṇita* of Bhāskarācārya II (c.1150); V. G. Apte (ed.), Pune 1930; T. V. Radha Krishna Sastry (ed.), Tanjore 1958; Biharilal Vasistha (ed.), Jammu 1982.
18. *Śiromaṇiprakāśa* of Gaṇeśa (c.1600) on *Siddhāntaśiromaṇi* of Bhāskarācārya II (c.1150), *Grahagaṇitādhyaya,* V. G. Apte (ed.), 2 Vols. Pune 1939, 1941.

19. *Gūḍhārthaprakāśa* of Raṅganātha (c.1603) on *Sūryasiddhānta*, Jivananda Vidyasagara (ed.), Calcutta 1891; Reprint, Varanasi 1990.
20. *Vāsanāvārttika*, commentary of Nṛsiṁha Daivajña (c.1621) on *Vāsanābhāṣya* of Bhāskarācārya II, on his own *Siddhāntaśiromaṇi* (c.1150), Muralidhara Chaturveda (ed.), Varanasi 1981.
21. *Marīci* of Munīśvara (c.1630) on *Siddhāntaśiromaṇi* of Bhāskarācārya (c.1150), *Madhyamādhikara*, Muralidhara Jha (ed.), Varanasi 1908; *Grahagaṇitādhyaya,* Kedaradatta Joshi (ed.), 2 vols. Varanasi 1964; *Golādhyāya*, Kedaradatta Joshi (ed.), Delhi 1988.
22. *Āśayaprakāśa* of Munīśvara (c.1646) on his own *Siddhāntasārvabhauma*, *Gaṇitādhyaya* Chs. I-II, Muralidhara Thakura (ed.), 2 Vols, Varanasi 1932, 1935; Chs. III-IX, Mithalal Ojha (ed.), Varanasi 1978.
23. *Śeṣavāsanā* of Kamalākarabhaṭṭa (c.1658) on his own *Siddhāntatattvaviveka*, Sudhakara Dvivedi (ed.), Varanasi1885; Reprint, Varanasi 1991.
24. *Sauravāsanā* of Kamalākarabhaṭṭa (c.1658) on *Sūryasiddhānta*, Chs. I-X, Srichandra Pandeya (ed.), Varanasi 1991.
25. *Gaṇitayuktayaḥ*, Tracts on Rationale in Mathematical Astronomy by various Kerala Astronomers (c.16th-19th century), K.V. Sarma, Hoshiarpur 1979.

APPENDIX II: UPAPATTI OF THE KUṬṬAKA PROCESS

The kuṭṭaka process

The *kuṭṭaka* process for solving linear indeterminate equations has been known to Indian mathematicians at least since the time of Āryabhaṭa (c.499 AD). Consider the first order indeterminate equation of the form

$$(ax \pm c)/b = y$$

Here a, b, c are given positive integers, and the problem is to find integral values of x, y that satisfy the above equation; a is called the *bhājya* (dividend), b the *bhājaka* or *hāra* (divisor), c the *kṣepa* (interpolator). The *kṣepa* is said to be *dhana* (additive) or *ṛṇa* (subtractive) depending on whether the 'plus' or 'minus' sign is taken in the above equation. The numbers to be found, x, is called the *guṇaka* (multiplier) and y the *labdhi* (quotient).

The process of solution of the above equation is referred to as the *kuṭṭaka* process. *kuṭṭaka* or *kuṭṭākāra* (translated as 'pulveriser') is the name for the *guṇaka* (multiplier) x. Kṛṣṇa Daivajña explains:

> *Kuṭṭaka* is the *guṇaka*; for, multiplication is referred to by terms (such as *hanana, vadha, ghāta,* etc.), which have connotation of "injuring", "killing" etc. By etymology and usage (*yogarūḍhi*), this term (*kuṭṭaka*) refers to a special multiplier. That number, which when multiplied by the given *bhājya* and augmented or diminished by the given *kṣepa* and divided by the given *hāra*, leaves no reminder, is called the *kuṭṭaka* by the ancients. [44]

The procedure for solution of the above equation is explained as follows by Bhāskarācārya in his *Bījagaṇita*; the relevant verses are 1-5 of the *Kuṭṭakādhyāya*:[45]

1. In the first instance, as preparatory to carrying out the *kuṭṭaka* process (or for finding the *kuṭṭaka*), the *bhājya, hāra* and *kṣepa* are to be factored by whatever number possible. If a number, which divides both the *bhājya* and *hāra*, does not divide the *kṣepa*, then the problem is an ill-posed problem.

2. When the *bhājya* and *hāra* are mutually divided, the last remainder is their *apavartana* or *apavarta* (greatest common factor). The *bhājya* and *hāra* after being divided by that *apavarta* will be characterised as *dṛḍha* (firm or reduced) *bhājya* and *hāra*.

3. Divide mutually the *dṛḍha-bhājya* and *hāra*, until unity becomes the remainder in the dividend. Place the quotients [of this mutual division] one below the other, the *kṣepa* below them and finally zero at the bottom.

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44 *Bījapallavam* commentary on *Bījagaṇita*, cited above, p.86.

45 *Bījapallavam* commentary on *Bījagaṇita*, cited above, p.86-89.

4. [In this *vallī*] the number just above the penultimate number is replaced by the product of that number with the penultimate number, with the last number added to it. Then remove the last term. Repeat the operation till only a pair of numbers is left. The upper one of these is divided [abraded] by the *dṛḍha-bhājya*, the remainder is the *labdhi*. The other (or lower) one being similarly treated with the (*dṛḍha*) *hāra* gives the *guṇa*.

5. This is the operation when the number of quotients [in the mutual division of *dṛḍha-bhājya* and *dṛḍha-hāra*] is even. If the number of quotients be odd then the *labdhi* and *guṇa* obtained this way should be subtracted from their abraders (*dṛḍha-bhājya* and *dṛḍha-hāra* respectively) to obtain the actual *labdhi* and *guṇa*.

An Example

Let us explain the above procedure with an example that also occurs later in the *upapatti* provided by Kṛṣṇa Daivajña.

Let *bhājya* be 1211, *hāra* 497 and *kṣepa* 21. The procedure above outlined is for additive *kṣepa* and the equation we have to solve is $1211x+21 = 497y$. The first step is to make *bhājya* and *hāra* mutually prime to each other (*dṛḍha*) by dividing them by their greatest common factor (*apavartāṅka*). The *apavartāṅka* is found by the process of mutual division (the so-called Euclidian algorithm) to be 7. Now dividing *bhājya, hāra* and *kṣepa* by this *apavartāṅka*, we get the *dṛḍha-bhājya*, *hāra* and *kṣepa* to be 173, 71 and 3 respectively. Thus, the problem is reduced to the solution of the equation $173x+3 = 71y$.

```
497)1211(2
     994
     217)497(2
         434
          63)217(3
             189
              28)63(2
                 56
                  7)28(4
                    28
                     0
```

Now by mutually dividing the *dṛḍha-bhājya* and *hāra*, the *vallī* of quotients (which are the same as before) is formed, below which are set the *kṣepa* 3 and zero. Following the procedure stated (in verse 4, above) we get the two numbers 117, 48. Now since the number of quotients is even, we need to follow the procedure (of verse 4 above) and get the *labdhi*, $y = 117$ and the *guṇa*, $x = 48$.

2	2	2	2	117	
2	2	2	48	48	48x2+21=117
3	3	21	21		21x2+6=48
2	6	6			3x6+3=21
3	3				2x3+0=6
0					

Now we shall present the *upapatti* of the above process as expounded by Kṛṣṇa Daivajña in his commentary on *Bījagaṇita*. For convenience of understanding we divide this long proof into several steps.

Proof of the fact that when the bhājya, hāra, and kṣepa are factored by the same number, there is no change in the labdhi and guṇa [46]

It is well known that whatever is the quotient (*labdhi*) of a given dividend and divisor, the same will be the quotient if both the dividend and divisor are multiplied or factored by the same number. In the present case, the given *bhājya* multiplied by some *guṇaka* and added with the positive or negative *kṣepa* is the dividend. The divisor is the given *hāra*. Now the dividend consists of two parts. The given *bhājya* multiplied by the *guṇaka* is one and the *kṣepa* is the other. If their sum is the dividend and if the dividend and divisor are both factored by the same number then there is no change in the *labdhi*. Therefore, that factor from which the divisor is factored, by the same factor is the dividend, which is resolvable into two parts, is also to be factored. Now the result is the same whether we factor the two parts and add or add the two parts and then factor the sum. Just as, if the dividend 27 is factored by 3 we get 9; alternatively the dividend is resolved into the parts 9, 18 which when factored by 3 give 3,6 and these when added gives the same factored dividend, viz., 9. In the same way in other instances also, if the dividend is resolved into two or many parts and these are factored and added, then the result will be the same as the factored dividend.

Therefore, when we factor the given *hāra*, then the given *bhājya* multiplied by the *guṇa* should also be factored and so also the *kṣepa*. Now *guṇa* being not known, the given *bhājya* multiplied by the *guṇa* will be another unknown, whose factoring is not possible; still, if the given *bhājya* is factored and then multiplied by the *guṇa*, then we get the same result as factoring that part of the dividend which is gotten by multiplying the given *bhājya* by *guṇa*. For, it does not make a difference whether we factor first and then multiply or whether we multiply first and then factor. Thus, just as the given *bhājya* multiplied by the *guṇa* will become one part in the resolution of the dividend, in the same way the factored *bhājya* multiplied by the same *guṇa* will become one part in the resolution of the factored dividend. The factored *kṣepa* will be the second part. In this manner, the *bhājya*, *hāra* and *kṣepa* all un-factored or factored will lead to no difference in the *guṇa* and *labdhi*, and hence for the sake of *laghava* (felicity of computation) it is said that 'the *bhājya, hāra* and *kṣepa* have to be factored…[verse 1]'. We will discuss whether the factoring is necessary or not while presenting the *upapatti* of 'Divide-mutually the *dṛḍha-bhājya* and *hāra*... [verse 3]'.

Proof of the fact that if a number, which divides both bhājya and hāra, does not divide the kṣepa, then the problem is ill-posed[47]

Now the *upapatti* of *khilatva* or ill-posed-ness: Here, when the divisor and dividend are factored, even though there is no difference in the quotient, there is always a

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[46] *Bījapallavam* commentary on *Bījagaṇita*, cited above, p.89-90.

[47] *Bījapallavam* commentary on *Bījagaṇita*, cited above, p. 90-91.

change in the remainder. The remainder obtained when (divisor and dividend) are factored, if multiplied by the factor, will give us the remainder for the original (un-factored) divisor and dividend. For instance, let the dividend and divisor be 21, 15; these factored by 3 give 7, 5. Now if the dividends are multiplied by 1 and divided by the respective divisors, the remainders are 6, 2; when dividends are multiplied by 2 the remainders are 12, 4; when multiplied by 3 they are 3, 1; when multiplied by 4 they are 9, 3; when multiplied by 5, they are 0, 0. If we multiply by 6, 7 etc. we get back the same sequence of remainders. Therefore, if we consider the factored divisor, 5, the remainders are 0, 1, 2, 3, 4 and none other than these. If we consider the un-factored divisor 15, the remainders are 0, 3, 6, 9, 12 and none other than these. Here, all the remainders have the common factor 3.

Now, let us consider the *kṣepa*. When the *guṇaka* (of the given *bhājya*) is such that we have zero remainder (when divided by the given *hāra*), then (with *kṣepa*) we will have zero remainder only when the *kṣepa* is zero or a multiple of *hāra* by one, two, etc., and not for any other *kṣepa* … For all other *guṇakas* which leave other remainders, (when multiplied by *bhājya* and divided by *hāra*) then (with *kṣepa*) we have zero remainder when *kṣepa* is equal to the *śeṣa* (remainder) or *hāra* diminished by *śeṣa*, depending on whether *kṣepa* is additive or subtractive and not for any other *kṣepa*, unless it is obtained from the above by adding *hāra* multiplied by one, two, etc. Thus, in either case of *kṣepa* being equal to the *śeṣa* or *hāra* diminished by the *śeṣa*, since *kṣepa* will be included in the class of *śeṣas* discussed in the earlier paragraph, the *kṣepa* will have the same *apavarta* or factor (that *bhājya* and *hāra* have). This will continue to be the case when we add any multiple of *hāra* to the *kṣepa*. Thus we do not see any such *kṣepa* which is not factorable by the common factor of the *bhājya* and *hāra*. Therefore, when the *kṣepa* is not factorable in this way, with such a *kṣepa* a zero remainder can never be obtained (when *bhājya* is multiplied by any *guṇa* and divided by *hāra* after adding or subtracting the *kṣepa* to the above product); because the *kṣepas* that can lead to zero remainder are restricted as discussed above. With no more ado, it is indeed correctly said that the problem itself is ill-posed when the *kṣepa* is not divisible by the common factor of *bhājya* and *hāra*.

Rationale for the procedure for finding apavartāṅka – the greatest common factor[48]

Now the rationale for the procedure for finding the *apavartāṅka* (greatest common factor). Here the *apavartāṅka* is to be understood as the factor such that when the divisor and dividend are factored by this, no further factorisation is possible. That is why they are said to be *dṛḍha* (firm, prime to each other) when factored by this *apavartāṅka*. Now the procedure for finding that: When dividend and divisor are equal, the greatest common factor (G.C.F.) is equal to them as is clear even to the dull-witted. Only when they are different will this issue become worthy of investigation. Now consider 221, 195. Between them the smaller is 195 and the G.C.F. being its divisor cannot be larger than that. The G.C.F. will be equal to the smaller if the larger number is divisible by the smaller number, i.e., leaves no remainder when divided. When the remainder is 26, then the G.C.F. cannot be equal to the smaller number 195, but will be smaller than that. Now let us look into that.

48 *Bījapallavam* commentary on *Bījagaṇita*, cited above, p.91-92.

The larger number 221 is resolvable into two parts; one part is 195 which is divisible by the smaller number and another part is 26, the remainder. Now among numbers less than the smaller number 195, any number which is larger than the remainder 26, cannot be the G.C.F.; for, the G.C.F. will have to divide both parts to which the large number 221 is resolved. Now the remainder part 26, itself, will be the G.C.F., if the smaller number 195 were divisible by 26. As it is not, the G.C.F. is smaller than the remainder 26. Now let us enquire further.

The smaller number 195 is resolvable into two parts; 182 which is divisible by the first remainder 26 and the second remainder 13. Now, if a number between the earlier remainder 26 and the second remainder 13 is a G.C.F., then that will have to somehow divide 26, and hence the part 182; but there is no way in which such a number can divide the other part 13 and hence it will not divide the smaller number 195.

Thus, among numbers less than the first remainder 26, the G.C.F. can be at most equal to the second remainder 13. That too only if when the first remainder 26 is divided by the second remainder 13 there is no remainder… Now when the first remainder divided by the second remainder leaves a (third) remainder, then by the same argument, the G.C.F. can at most be equal to the third remainder. And, by the same *upapatti*, when it happens that the previous remainder is divisible by the succeeding remainder then that remainder is the greatest common factor. Thus is proved "When *bhājya* and *hāra* are mutually divided, the last (non-zero) remainder is their *apavarta*… (verse 2)".

Rationale for the kuṭṭaka process when the kṣepa is zero[49]

When there is no *kṣepa*, if the *bhājya* is multiplied by zero and divided by *hāra* there is no remainder and hence zero itself is both *guṇa* and *labdhi*; or if we take *guṇa* to be equal to *hāra*, then since *hāra* is divisible by *hāra* we get *labdhi* equal to *bhājya*. Therefore, when there is no *kṣepa*, then zero or any multiple of *hāra* by a desired number will be the *guṇa* and zero or the *bhājya* multiplied by the desired number will be the *labdhi*. Thus here, if the *guṇa* is increased by an amount equal to *hāra* then the *labdhi* will invariably be increased by an amount equal to *bhājya* …

Now even when the *kṣepa* is non-zero, if it be equal to *hāra* or a multiple of *hāra* by two, three, etc. then the *guṇa* will be zero, etc. as was stated before. For, with such a *guṇa*, there will be a remainder (when divided by *hāra*) only because of *kṣepa*. But if *kṣepa* is also a multiple of *hāra* by one, two, etc. how can there arise a remainder? Thus for such a *kṣepa*, the *guṇa* is as stated before. In the *labdhi* there will be an increase or decrease by an amount equal to the quotient obtained when the *kṣepa* is divided by *hāra*, depending on whether *kṣepa* is positive or negative…

[49] *Bījapallavam* commentary on *Bījagaṇita*, cited above, p.92-93.

Rationale for the kuṭṭaka process when kṣepa is non-zero[50]

Now when the *kṣepa* is otherwise: The *upapatti* is via resolving the *bhājya* into two parts. The part divisible by *hāra* is one. The remainder is the other. When *bhājya* and *hāra* are 16, 7 the parts of *bhājya* as stated are 14, 2. Now since the first part is divisible by *hāra*, if it is multiplied by any *guṇa* it will still be divisible by *hāra*. Now if the given *kṣepa* when divided by the second part leaves no remainder, then the quotient obtained in this division is the *guṇa* (in case the *kṣepa* is subtractive). For, when this *guṇa* multiplies the second part of *bhājya* and *kṣepa* is subtracted then we get zero. Now if the *kṣepa* is not divisible by the second part, then it is not simple to find the *guṇa* and we have to take recourse to other procedure.

When the *bhājya* is divided by *hāra*, if 1 is the remainder, then the second part is also 1 only. Then whatever be the *kṣepa*, if this remainder is multiplied by *kṣepa* we get back the *kṣepa* and so we can apply the above procedure and *guṇa* will be equal to *kṣepa*, when *kṣepa* is subtractive, and equal to *hāra* diminished by *kṣepa*, when the *kṣepa* is additive. In the latter case, when the *guṇa* multiplies the second part of *bhājya* we get *hāra* diminished by *kṣepa*. When we add *kṣepa* to this we get *hāra*, which is trivially divisible by *hāra*. The *labdhi* will be the quotient, obtained while *bhājya* is divided by *hāra*, multiplied by *guṇa* in the case of subtractive *kṣepa* and this augmented by 1 in the case of additive *kṣepa*.

Now when *bhājya* is divided by *hāra* the remainder is not 1, then the procedure to find the *guṇa* is more complicated. Now take the remainder obtained in the division of *bhājya* by *hāra* as the divisor and *hāra* as the dividend. Now also if 1 is not the remainder then the procedure for finding the *guṇa* is yet more difficult. Now divide the first remainder by the second remainder. If the remainder is 1, then if the first remainder is taken as the *bhājya* and the second remainder is *hāra*, we can use the above procedure to get the *guṇa* as *kṣepa* or *hāra* diminished by *kṣepa*, depending on whether the *kṣepa* is additive or subtractive. But if the remainder is larger than 1 even at this stage, then the procedure to find *guṇa* is even more complicated. Therefore when we go on doing mutual division, we want to arrive at remainder 1 at some stage. But how can that be possible if *bhājya* and *hāra* have a common factor, for the ultimate remainder in mutual division is the greatest common factor. Now if we factor the *bhājya* and *hāra* by the *apavartāṅka* (greatest common factor) then the remainders will also factored by that, and the final remainder will be unity. This is why it is necessary to first reduce both *bhājya* and *hāra* by their greatest common factor.

Now, even when the penultimate remainder considered as a *bhājya* gives unity as the remainder when divided by the next remainder (considered as *hāra*) and from that a corresponding *guṇa* can be obtained, how really is one to find the *guṇa* appropriate to the originally specified *bhājya*. That is to be found by *vyasta-vidhi*, the reverse process or the process of working backwards. Now let the *bhājya* be 1211, *hāra* 497 and *kṣepa* 21. If *bhājya* and *hāra* are mutually divided, the final remainder (or their G.C.F.) is 7. Factoring by this, the reduced *bhājya*, *hāra* and *kṣepa* are 173, 71 and 3 respectively. Now by mutual division of these *dṛḍha-bhājya* and *hāra*, we get the *vallī* (sequence)

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50 *Bījapallavam* commentary on *Bījagaṇita*, cited above, p.93-98.

of quotients 2, 2, 3, 2 and remainders 31, 9, 4, 1 and the various *bhājyas* and *hāras* as follows:

bhājya	173	71	31	9
hāra	71	31	9	4

Now in the last *bhājya* 9, there are two parts: 8 which is divisible by *hāra* 4 and remainder 1. Using the procedure stated above, the *guṇa* will be the same as the *kṣepa* 3, for the case of subtractive *kṣepa*. The quotient 2 (of the division of the last *bhājya* 9 by the *hāra* 4) multiplied by this *guṇa* 3 will give the *labdhi* 6. It is for this reason it is said that, "place the quotients one blow the other, the *kṣepa* below them and finally zero at the bottom… (verse 3)." Here the "last quotient multiplied by below… (verse 4)" gives (the changed *vallī* as shown):

2
2
3
2
3 *kṣepa*
0

2
2
3
6 *labdhi*
3 *guṇa*

Now keeping the same *kṣepa*, we will discuss what will be the *guṇa* for the earlier pair of *bhājya* and *hāra* (given by) 31, 9. Here also the parts (to which the *bhājya* is to be resolved) as stated above are 27, 4. Now the first part, whatever be the number it is multiplied by, is divisible by *hāra*. Thus it is appropriate to look at the second part while considering the *guṇa* and *labdhi*. Thus we have the pair of *bhājya* and *hāra* 4, 9. This is only the previous pair (of 9, 4) considered with the *bhājya* and *hāra* interchanged and this leads to an interchange of the *guṇa* and *labdhi* also. This can be seen as follows.

The *bhājya* 9, multiplied by *guṇa* 3 leads to 27 (and this) diminished by *kṣepa* 3 gives 24 (and this) divided by *hāra* 4 gives the *labdhi* 6. Now by inverse process, this *labdhi* 6 used as a *guṇa* of the new *bhājya* 4, gives 24 (and this) augmented by *kṣepa* 3 gives 27 (and this) is divisible by the new *hāra* 9; and hence 6, the *labdhi* for the last pair (of *bhājya* and *hāra*) is the *guṇa* for the present. The *labdhi* (considering the second part alone) is 3, the *guṇa* for the last pair. But for the given *bhājya* (31), the *labdhi* for the earlier part (27) multiplied by the *guṇa* is to be added. The *guṇa* is the penultimate entry (6) in the *vallī*. The *labdhi* for the first part is the quotient (3) set down above that. And these two when multiplied will give the *labdhi* (18) for the first part. This is to be added to the *labdhi* for the second part which is 3 the last entry in the *vallī*. Thus we get the new *vallī*.

2
2
3
6 *guṇa*
3 *labdhi*

2
2
21 *labdhi*
6 *guṇa*
3 *labdhi*

The last entry 3 is no long relevant and omitting that we get the *vallī*. So it is said "multiply the penultimate number by the number just above and add the earlier term. Then reject the lowest… (verse 4)". Thus for the pair 31, 9 we have obtained by the inverse process (*vyasta-vidhi*) the *labdhi* and *guṇa* 21, 6 for additive (*kṣepa*).

2
2
21 *labdhi*
6 *guṇa*

Now for the still earlier pair of *bhājya* and *hāra*, namely 71, 31 and with the same *kṣepa*, let us enquire about the *guṇa*. Here again (the *bhājya* is divided into) parts 62, 9 as stated above, and keeping the first part aside we get the pair of *bhājya* and *hāra*, 9, 31. Again, since we have only interchanged the earlier *bhājya* and *hāra*, the same should happen to *labdhi* and *guṇa*. Thus we have as *guṇa* and *labdhi* 21, 6. Here also the *labdhi* of the first part is to be multiplied by the *guṇa*. The penultimate entry in the *vallī*, 21, which is now the *guṇa* is multiplied by the 2 which is above it and which is the *labdhi* of the first part (62), and to the result 42 is added the *labdhi* 6 of the second part (9), and thus we get the total *labdhi* 48.

The last entry of the *vallī* as shown is removed as before, and we get the *vallī*. Thus by the inverse process we get for the pair of *bhājya* and *hāra* 71, 31, and for a subtractive *kṣepa*, the *labdhi* and *guṇa* 48, 21.

2	
48	*labdhi*
21	*guṇa*

Now the enquiry into the *guṇa* associated with the yet earlier pair of *bhājya* and *hāra*, 173, 71. Here also splitting (the *bhājya*) into two parts 142, 31 as stated before, we get the *bhājya* and *hāra* 31, 71. Here again, we only have an interchange of *bhājya* and *hāra* from what we discussed before and so by interchanging the *labdhi* and *guṇa* as also the (status of additivity or subtractivity of the) *kṣepa*, we get the *labdhi* and *guṇa* 21, 48 for additive *kṣepa*. Here again to get the *labdhi* of the first part, the penultimate (entry in the *vallī*) 48 is multiplied by the entry 2 above it to get 96.

To get the total *labdhi*, the last entry 21 is added to get 117. Removing the last entry of the *vallī* which is no longer of use, we get the *vallī* as shown.

117	*labdhi*
48	*guṇa*

Thus for the main (or originally intended) pair of *bhājya* and *hāra* 173, 71 and with additive *kṣepa* 3, the *labdhi* and *guṇa* obtained are 117, 48. Therefore it is said 'Repeat the operation till only two numbers are left... (verse [4])"

Except for the last *bhājya*, in all *bhājyas*, while getting the *labdhi* for the first part, the *guṇa* will be penultimate (in the *vallī*) and hence it is said that the penultimate is multiplied by the number above. That is to be added to the last number which is the *labdhi* for the second part (of the *bhājya*). For the last *bhājya*, the last entry is the *guṇa* and there is no *labdhi* for the second part. Hence Ācārya has instructed the inclusion of zero below in the end (of the *vallī*) so that the procedure is the same all through. Thus are obtained the *labdhi* and *guṇa* 117, 48.

Now, it has been seen earlier itself that if we increase *guṇa* by *hāra*, then *labdhi* will get increased by *bhājya*; and by the same argument, if the *guṇa* is diminished by *hāra*, the *labdhi* will get diminished by *bhājya*. Hence when the *guṇa* is larger than *hāra*, then once, twice or, whatever be the number of times it may be possible, the *hāra* is to be subtracted from that *guṇa* so that a smaller *guṇa* is arrived at. The *labdhi* is (reduced by a multiple of *bhājya*) in the same way. Hence it is said "The upper one of these is divided [abraded] by the *dṛḍha-bhājya*, the remainder is the *labdhi*. The other (or lower) one being similarly treated with the (*dṛḍha*) *hāra* gives the *guṇa* (verse 4)" (Ācārya) also emphasises the above principle (in a) later (verse of *Bījagaṇita*): "The number of times that the *guṇa* and labdhi are reduced should be the same." If *guṇa* is reduced by *hāra* once, then the labdhi cannot be diminished by twice the *bhājya* and so on.

Labdhi and guṇa for even and odd number of quotients[51]

If it were asked how we are to know whether the *labdhi* and *guṇa*, as derived above for the main *bhājya*, correspond to additive or subtractive *kṣepa*; for (it may be said that) in the case of the last and penultimate *bhājyas*, it is not clear whether the *guṇa* is for additive or subractive *kṣepas*, we state as follows. For the last pair of *bhājya* and *hāra*, the *guṇa* was derived straightaway taking the *kṣepa* to be subtractive. Thus by the *vyasta-vidhi* (inverse process), for the penultimate pair the *guṇa* that we derived was for additive *kṣepa*. For the third pair, the *guṇa* that we derived was for subtractive *kṣepa*. It would be additive for the fourth and subtractive for the fifth pair. Now starting from the last pair, for each even pair, the *guṇa* derived would be for additive *kṣepa* and for each odd pair, it would be for subtractive *kṣepa*. Now for the main (or originally given) pair of *bhājya* and *hāra*, this even or odd nature is characterised by the even or odd nature of the number of quotients in their mutual division. Hence, if the number of quotients is even, then the *labdhi*, *guṇa* derived are for additive *kṣepa*. If they are odd then the *labdhi* and *guṇa* derived for the main (or originally given) *bhājya* and *hāra* are for subtractive *kṣepa*. Since the (Ācārya) is going to state a separate rule for subtractive *kṣepa*, here we should present the process for additive *kṣepa* only. Hence it is said, "what are obtained are (the *labdhi* and *guṇa*) when the quotients are even in number… (verse 5)".

When the number of quotients is odd, the *labdhi* and *guṇa* that are obtained are those valid for subtractive *kṣepa*. But what are required are those for additive *kṣepa*. Hence it is said that "If the number of quotients be odd then the *labdhi* and *guṇa* obtained this way should be subtracted from their abraders…(verse 5)." The rationale employed here is that the *guṇa* for subtractive *kṣepa*, if diminished from *hāra* will result in the *guṇa* for additive *kṣepa*.

This can also be understood as follows. Any *bhājya* which on being multiplied by a *guṇa* is divisible (without remainder) by its *hāra*, the same will hold when it is multiplied by the (two) parts of the *guṇa* and divided by the *hāra*. The *labdhi* will be the sum of the quotients. If there is a reminder when one of the partial products is divided by the *hāra*, the other partial product will be divisible by the *hāra* when it increased by the same remainder - or else the sum of the two partial products will not be divisible by the *hāra*.

Now if the *bhājya* is multiplied by a *guṇa* equal to *hāra* and then divided by the *hāra* it is clearly divisible and the *labdhi* is also the same as *bhājya*. Since the *guṇa* and *hāra* are the same in this case, the parts of the *guṇa* are the same as that of the *hāra*. For example if *bhājya* is 17, *hāra* 15 and *guṇa* is also 15, then *bhājya* multiplied by *guṇa* is 225 and divided by *hāra* gives *labdhi* 17. If the two parts of *guṇa* are 1, 14,

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51 *Bījapallavam* commentary on *Bījagaṇita*, cited above, p.98-99.

then the partial products are 17, 238. The first, if divided by *hāra*, leaves remainder 2. If we reduce this by the same *kṣepa* of 2, then it will be divisible, and *labdhi* will be 1. The other partial product, if increased by the same *kṣepa*, becomes 240 and will be divisible by the *hāra*. The *labdhi* will be 16. Or, if the parts of the *guṇa* are 2, 13, the partial products are 34, 221. The first when divided by *hāra* gives the remainder 4, and if reduced by that, it becomes 30 and will be divisible by the *hāra* and *labdhi* will be 2 and the partial *guṇa* 2. The other partial product 221, if increased by the same remainder, will be divisible by the *hāra* and the *labdhi* will be 15 and the partial *guṇa* 13. Or, if the parts of the *guṇa* are 3, 12, the partial products will be 51, 204. The first one when reduced by 6 and the second when increased by 6, will be divisible. Thus for *kṣepa* of 6, the *guṇas* when it is additive and subtractive are respectively the parts 12, 3. The *labdhis* are correspondingly 14, 3.

Hence the Ācārya states (later in *Bījagaṇita*) "The *guṇa* and *labdhi* obtained for additive *kṣepa*, when diminished by their abraders, will result in those for negative *kṣepa* ". Thus the procedure for arriving at *guṇa* and *labdhi* as outlined in the text starting with "Divide mutually…" and ending with "…to give the actual *labdhi* and *guṇa* " (i.e., verses 3 to 5) has been demonstrated (*upapannam*).

APPENDIX III: MATHEMATICAL TOPICS AND PROOFS IN YUKTIBHĀṢĀ (C.1530)

In this Appendix we shall present an outline of the topics and proofs contained in the Mathematics part of the celebrated Malayalam text *Yuktibhāṣā* [52] of Jyeṣṭhadeva (c.1530). This part is divided into seven Chapters, of which the last two, entitled *Paridhi and Vyāsa* (Circumference and Diameter) and *jyānayanam* (Computation of Sines), contain many important results concerning infinite series and fast convergent approximations for π and the trigonometric functions. In the preamble to his work, Jyeṣṭadeva states that his work closely follows *Tantrasaṅgraha* of Nīlakaṇṭha (c.1500) and gives all the mathematics necessary for the computation of planetary motions. The proofs expounded by Jyeṣṭhadeva have been reproduced (mostly in the form of Sanskrit verses–*kārikās)* by Śaṅkara Vāriyar in his commentaries *Yuktidīpikā*[53] on *Tantrasaṅgraha* and *Kriyākramakarī*[54] on *Līlāvatī*. Since the later work is considered to be written around 1535 A.D., the time of composition of *Yuktibhāṣā* may reasonably be placed around 1530 A.D.

In what follows we shall present an outline of the mathematical topics and proofs given in *Yuktibhāṣā*, following closely the order which they appear in the text.

I. *Parikarmāṇi* (Mathematical Operations)

Following *Tantrasaṅgraha* an exposition of all the mathematics necessary thereof:

Numbers, place value, the eight operations involving increase and decrease, addition and subtraction.

Multiplication: Methods of multiplication, representation of the product as a *ghāta-kṣetra* (rectangle), geometrical representation of different methods of multiplication involving adding, subtracting or factoring a number from the multiplicand, division.

Squaring: Algorithm for squaring, identifying the terms which occur at different odd and even places, other methods of squaring. Geometrical representation of the identity

$$(a+b)^2 = a^2 + b^2 + 2\,ab = 4\,ab + (a-b)^2$$

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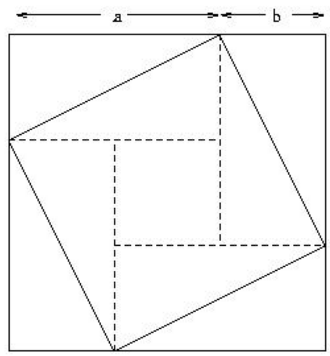
[52] *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530); *Gaṇitādhyāya*, Ramavarma Thampuran and A.R. Akhileswara Aiyer (eds.), Trichur 1947; K. Chandrasekharan (ed.), Madras 1953; Edited, along with an ancient Sanskrit version *Gaṇitayuktibhāṣā* and English Translation, by K.V.Sarma (in Press).

[53] *Yuktidīpikā* of Śaṅkara Vāriyar (c.1530) on *Tantrasaṅgraha* of Nīlakaṇṭha Somasutvan (c.1500), K.V. Sarma (ed.), Hoshiarpur 1977. At the end of each chapter of this work, Śaṅkara states that he is only presenting the material which has been well expounded by the great *dvija* of the Parakroḍha house, Jyeṣṭhadeva.

[54] *Kriyākramakarī* of Śaṅkara Vāriyar (c.1535) on *Līlāvatī* of Bhāskarācārya II (c.1150), K.V. Sarma (ed.), Hoshiarpur 1975.

Demonstration of the *bhujā-koṭi-karṇa-nyāya* that the square of the diagonal of a rectangle is the sum of the squares of the sides (Pythagoras Theorem): This involves consideration of the following figure.



Geometrical representation of the identity

$$a^2 - b^2 = (a + b)(a - b)$$

A demonstration of the result that the arithmetical progression of odd numbers 1, 3, 5,… add up to the squares of successive natural numbers.

Square-root: Process of extracting square-root as the inverse of the process of squaring. Root of sum and difference of two squares in a process of iteration.

II. *Daśapraśnam* (Ten Questions)

To find a, b, given any two of the five quantities $a + b$, $a - b$, ab, $a^2 + b^2$, $a^2 - b^2$.

III *Bhinnagaṇitam* (Mathematics of Fractions)

Nature of a fraction, *savarṇīkaraṇa* (reducing a set of fractions to same denomination – common denominator)
Addition, subtraction, multiplication and division of fractions, squares and square-roots.

IV *Trairāśikam* (Rule of Three)

Rule of three for the parts of a composite, inverse rule of three.
Almost all mathematical computations are pervaded by the *trairaśika-nyāya* and *bhujā-koṭi-karṇa-nyāya*.

V *Kuṭṭākārah* (Linear Indeterminate Equation)

Calculation of *ahargaṇa* (number of mean civil days elapsed since epoch), calculation of mean longitudes. Given the mean longitude of a planet, the corresponding *ahargaṇa* can be found by the *kuṭṭaka* process.

The *kuṭṭaka* process (as given in *Līlāvatī* 242-246). Demonstration of the *kuṭṭaka* process by a consideration of the example (*Līlāvatī* 247)

$$(221x + 65)/195 = y$$

How the process of mutual division (of the *bhājya* and *bhājaka*, 221 and 195) gives the *apavartana* (common factor) 13. Why there is no solution unless the *kṣepa* (here 65) is also divisible by the common factor, and then the equation is reduced to

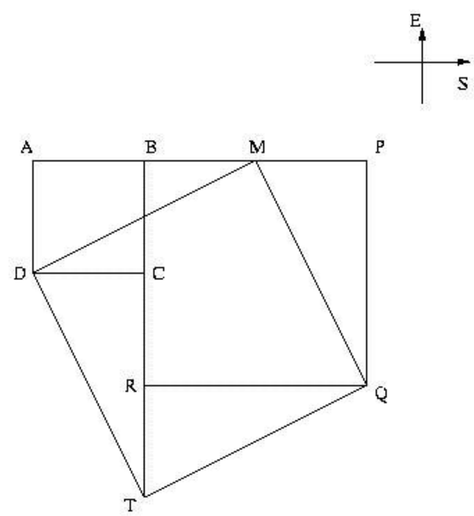
$$(17x + 5)/15 = y$$

Formation of the *vallī* (table of quotients of mutual division) and how the process of *vallī-upasaṁhāra* (sequence of reverse operations) leads step by step from solution of simpler problems ultimately to the solution of the original equation.

Application of *kuṭṭaka* for the calculation of *ahargaṇa* (number of mean civil days elapsed since epoch) and *bhagaṇas* (total number of completed revolutions) of mean Sun.

VI *Paridhi and Vyāsa* (Circumference and Diameter)

Another proof of bhujā - koṭi - karṇa - Nyāya



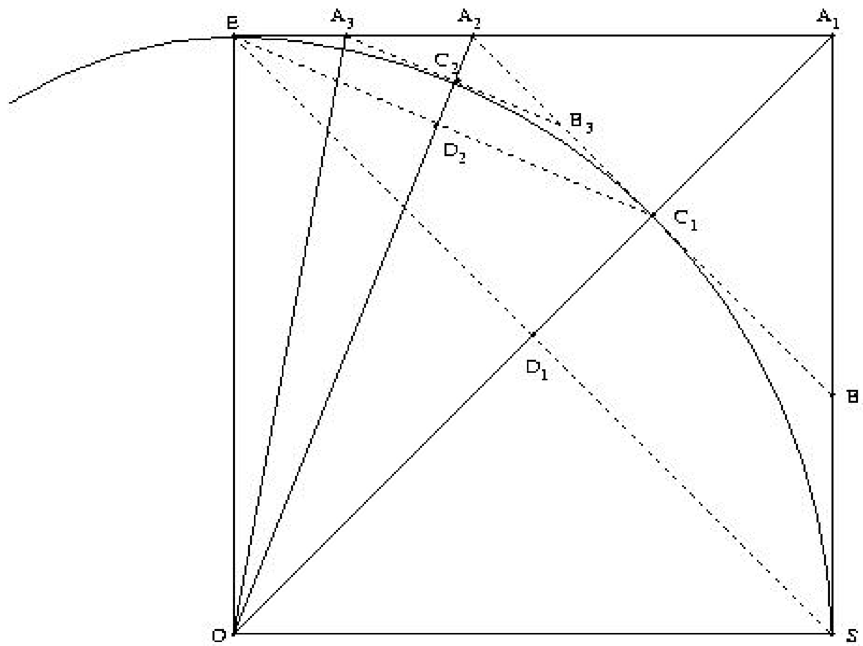
ABCD, square with its side equal to the *bhujā*, is placed on the north and the *koṭi*-square BRPQ is placed on the south, such that the eastern sides of both fall on the same line and the south side of the *bhujā* -square lies along the north side of the *koṭi*-square. Let *bhujā* be smaller than the *koṭi*.

Mark M on AP such that AM = BP = *koṭi*.
Hence MP = AB = *bhujā* and MD = MQ = *karṇa*.
Cut along MD and MQ, such that the triangles AMD and PMQ just cling at D, Q respectively. Turn them around to coincide with DCT and QRT. Therefore

karṇa-square MDTQ = *bhujā*-square ABCD + *koṭi*-square BPQR

Successive approximations to the circumference of a circle

By computing successively the perimeters of circumscribing square, octagon, regular polygon of sides 16, 32 etc using the following method:



EA_1SO is the quadrant of the square circumscribing the circle. EA_1 is half the side of the circumscribing square. Let the *karṇa* OA_1 meet the circle at C_1 and ES meet OA_1 at D_1. Draw the tangent $A_2C_1B_2$ parallel to ES to meet EA_1 at A_2. Then EA_2 is half the side of the circumscribing regular octagon. Let OA_2 meet the circle at C_2 and EC_1 meet OA_2 at D_2. Draw $A_3C_2B_3$ parallel to EC_1 to meet EA_1 at A_3. Then EA_3 is half the side of the regular polygon of 16 sides circumscribing the circle, and so on.

$EA_n = b_n/2$, where b_n is the *bhujā* or side of a regular polygon of 2^{n+1} sides. $OA_n = k_n$ the corresponding *karṇa* and $A_nD_n = a_n$ the corresponding *ābādhā* in the triangle OEA_n.

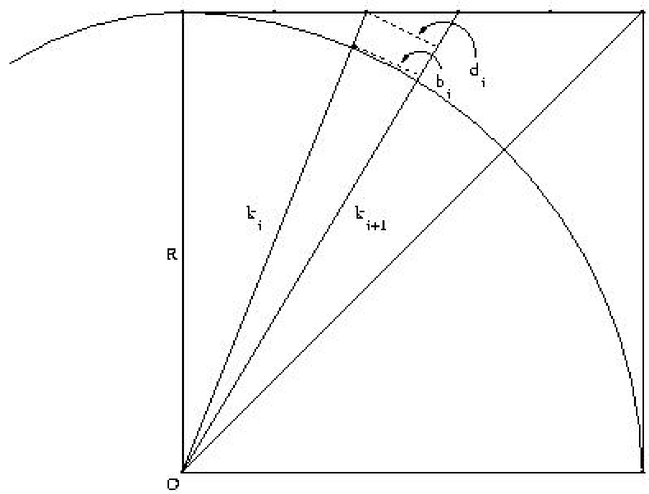
Now $b_1/2= R$. Given b_n proceed as follows to calculate b_{n+1}:
$OA_n = k_n$ is obtained by $k_n^2 = R^2 + (b_n/2)^2$
$A_nD_n = a_n$ is obtained by $a_n = (½)[k_n - \{R^2 - (b_n/2)^2\}/k_n]$

Finally b_{n+1} is obtained by

$(b_n - b_{n+1})/2 = A_nA_{n+1} = EA_n \,.\, A_nC_n / A_nD_n = (b_n/2)(k_n - R)/a_n$

To obtain the circumference without calculating square-roots

Consider a quadrant of the circle, inscribed in a square and divide a side of the square, which is tangent to the circle, into a large number of equal parts. The more the number of divisions the better is the approximation to the circumference.



$C/8$ (one eighth of the circumference) is approximated by the sum of the *jyārdhas* (half-chords) b_i of the arc-bits to which the circle is divided by the *karṇas* which join the points which divide tangent are joined to the centre of the circle. Let k_i be the length of these *karṇas*.

$$b_i = (R/k_i)d_i = (R/k_i)\,[(R/n)\,R/k_{i+1}] = (R/n)\,R^2/\,k_i\,k_{i+1}$$

Hence

$$\pi/4 = C/8R \approx (1/n)\sum_{i=0}^{n-1} R^2/k_i\,k_{i+1} \approx (1/n)\sum_{i=0}^{n-1} (R/k_i)^2$$

$$\pi/4 \approx (1/n)\sum_{i=0}^{n-1} R^2/[R^2 + i^2\,(R/n)^2]$$

Series expansion of each term in the right hand side is obtained by iterating

$$a/b = a/c - (a/b)\,(b\text{-}c)/c$$

which leads to

$$a/b = a/c - (a/c)\,(b\text{-}c)/c + (a/c)\,((b\text{-}c)/c)^2 + \ldots$$

This (binomial) series expansion is also justified later by showing how the partial sums in the following series converge to the result.

$$100/10 = 100/8 - (100/8)(10\text{-}8)/8 + 100/8\,[(10\text{-}8)/8]^2 - \ldots.$$

Thus

$$\pi/4 \quad \approx \quad 1 - (1/n)^3 \sum_{i=1}^{n} i^2 + (1/n)^5 \sum_{i=1}^{n} i^4 - \ldots.$$

When n becomes very large, this leads to the series given in the rule of Mādhava, *Vyāse vāridhnihate...* [55]

$$C/4D = \pi/4 = 1 - 1/3 + 1/5 - \ldots.$$

Sama-ghāta-saṅkalita – Sums of powers of natural numbers

In the above derivation, the following estimate was been employed for the *sama-ghāta-saṅkalita* of order k, for large n:

$$S_n^{(k)} = 1^k + 2^k + 3^k + \ldots. + n^k \approx n^{k+1}/(k+1)$$

This is proved first for the case of *mūla- saṅkalita*

$$S_n^{(1)} = 1 + 2 + 3 + \ldots. + n$$

$$= [n\text{-}(n\text{-}1)] + [n\text{-}(n\text{-}2)] + \ldots. + n = n^2 - S_{n-1}^{(1)}$$

Hence, for large n,

$$S_n^{(1)} \approx n^2/2$$

Then, for the *varga-saṅkalita* and the *ghana-saṅkalita* the following estimates are proved for large n:

$$S_n^{(2)} = 1^2 + 2^2 + 3^2 + \ldots. + n^2 \approx n^3/3$$

$$S_n^{(3)} = 1^3 + 2^3 + 3^3 + \ldots. + n^3 \approx n^4/4$$

In each case, the derivation is based on the result

$$n\, S_n^{(k-1)} - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + \ldots. + S_1^{(k-1)}$$

Now if we have already shown that $S_n^{(k-1)} \approx n^k/k$, then

$$n\, S_n^{(k-1)} - S_n^{(k)} \approx (n\text{-}1)^k/k + (n\text{-}2)^k/k + \ldots..$$

$$\approx S_n^{(k)}/k$$

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[55] This result is attributed to Mādhava by Śaṅkara Vāriyar in *Kriyākramakarī*, cited earlier, p.379; see also, *Yuktidīpikā*, cited earlier, p.101.

Hence, for the general *sama-ghāta-saṅkalita*, we get the estimate

$$S_n^k \approx n^{k+1}/(k+1)$$

Vāra-saṅkalita – Repeated sums

The *vāra-saṅkalita*, or repeated sums, are defined as follows:

$$V_n^{(1)} = S_n^{(1)} = 1 + 2 + \ldots + n$$

$$V_n^{(r)} = V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)}$$

It is shown that, for large n

$$\Sigma_n^{(r)} \approx n^{r+1}/(r+1)!$$

Cāpīkaraṇa – Determination of the arc

This can be done by the series given by the rule[56] *Iṣṭajyātrijyayorghātāt…*, which is derived in the same way as the above series for $C/8$:

$$R\theta = R\,(\sin\theta/\cos\theta) - (R/3)\,(\sin\theta/\cos\theta)^3 + (R/5)\,(\sin\theta/\cos\theta)^5 - \ldots$$

It is said that $\sin\theta \leq \cos\theta$ is a necessary condition for the terms in the above series to progressively lead to the result. Using the above for $\theta = \pi/6$, leads to the following:

$$C = (12D^2)^{1/2}\,(1 - 1/3.3 + 1/3^2.5 - 1/3^3.7 + \ldots)$$

Antya-saṁskāra – Correction term to obtain accurate circumference

Let us set

$$C/4D = \pi/4 = 1 - 1/3 + 1/5 - \ldots \pm 1/(2n-1) - (\pm)1/a_n$$

Then the *saṁskāra-hāraka* (correction divisor), a_n will be accurate if

$$1/a_n + 1/a_{n+1} = 1/(2n+1)$$

This leads to the successive approximations:[57]

$$\pi/4 \approx 1 - 1/3 + 1/5 - \ldots \pm 1/(2n-1) - (\pm)\,1/4n$$

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56 See, for instance, *Kriyākramakarī*, cited earlier, p.385; *Yuktidīpikā*, cited earlier, p.95-96.

57 These are attributed to Mādhava in *Kriyākramakarī*, cited earlier, p.279; also cited in *Yuktidīpikā*, cited earlier, p.101.

$$\pi/4 \approx 1 - 1/3 + 1/5 - \ldots \pm 1/(2n\text{-}1) - (\pm)\ 1/[4n + (4/4n)]$$

$$= 1 - 1/3 + 1/5 - \ldots \pm 1/(2n\text{-}1) - (\pm)\ n/(4n^2 + 1)$$

Later at the end of the chapter, the rule,[58] *Ante samasaṁkhyādalavargaḥ..*, is cited as the *sūkṣmatara-saṁskāra*, more accurate correction: [59]

$$\pi/4 \approx 1 - 1/3 + 1/5 - \ldots \pm 1/(2n\text{-}1) - (\pm)\ (n^2+1)/(4n^3 + 5n)$$

Transformation of series

The above correction terms can be used to transform the series for the circumference as follows::

$$C/4D = \pi/4 = [1 - 1/a_1] - [1/3 - 1/a_1 - 1/a_2] + [1/5 - 1/a_2 - 1/a_3] \ldots$$

It is shown that, using the second order correction terms, we obtain the following series given by the rule[60] *Samapañcāhatayoḥ…*

$$C/16D = 1/(1^5 + 4.1) - 1/(3^5 + 4.3) + 1/(5^5 + 4.5) - \ldots$$

It is also noted that by using merely the lowest order correction terms, we obtain the following series given by the rule[61] *Vyāsad vāridhinihatāt…*

$$C/4D = 3/4 + 1/(3^3 - 3) - 1/(5^3 - 5) + 1/(7^3 - 7) - \ldots$$

Other series expansions

It is further noted that to calculate the circumference one can also employ the following series as given in the rules[62] *Dvyādiyujāṁ vā kṛtayo…*and *Dvyādeś-caturādervā…*

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58 *Kriyākramakarī*, cited earlier p.390, *Yuktidīpikā*, cited earlier, p.103.

59 These correction terms are successive convergents of the continued fraction

$$1/a_n = 1/4n+ \quad 4/4n+ \quad 16/4n+ \quad \ldots$$

By using the third order correction term after 25 terms in the series, we get the value of π correct to eleven decimal places, which is what is given in the rule *Vibudhanetragajāhihutāśana*…, attributed to Mādhava by Nīlakaṇṭha (see his *Āryabhaṭīyabhāṣya*, *Gaṇitapāda*, K.Sambasiva Sastri (ed.), Trivandrum 1930, p. 56; see also *Kriyākramakarī*, cited earlier, p. 377):

$$\pi \approx 2827433388233/900000000000 = 3.141592653592222\ldots$$

60 *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā*, cited earlier, p.102.

61 *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā*, cited earlier, p.102.

62 *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā*, cited earlier, p.103.

$C/4D = 1/2 + 1/(2^2 - 1) - 1/(4^2 - 1) + 1/(6^2 - 1) - \ldots$

$C/8D = 1/(2^2 - 1) + 1/(6^2 - 1) + 1/(10^2 - 1) - \ldots$

$C/8D = 1/2 - 1/(4^2 - 1) - 1/(8^2 - 1) - 1/(12^2 - 1) - \ldots$

For the first series, a correction term is also noted:

$$C/4D \approx 1/2 + 1/(2^2 - 1) - 1/(4^2 - 1) + 1/(6^2 - 1) - \ldots$$

$$\pm 1/((2n)^2 - 1) - (\pm)\ 1/[2(2n + 1)^2 + 4]$$

VII *Jyānayanam* (Computation of Sines)

Jyā, *koṭi* and *śara* – $R\sin x$, $R\cos x$ and $R\text{versin}\, x = R\,(1\text{-}\cos x)$

Construction of an inscribed regular hexagon with side equal to the radius, which gives $R\sin(\pi/6)$

The relations

$R\sin(\pi/2 - x) = R\cos x = R(1\text{-versin}\, x)$

$R\sin(x/2) = ½\,[(R\sin x)^2 + (R\text{versin}\, x)^2]^{1/2}$

Using the above relations several sines can be calculated starting from the following:

$R\sin(\pi/6) = R/2.$

$R\sin(\pi/4) = (R^2/2)^{1/2}.$

Accurate determination of the *paṭhita-jyā* (enunciated or tabulated sine values) when a quadrant of the circle is divided into 24 equal parts of 3˚45' = 225' each. This involves estimating successive sine differences.

To find the sines of intermediate values, a first approximation is

$R\sin(x + h) \approx R\sin x + h\, R\cos x$

A better approximation as stated in the rule[63] *Iṣṭadoḥkoṭidhanuṣoḥ*… is the following:

$R\sin(x + h) \approx R\sin x + (2/\Delta)\,(R\cos x - (1/\Delta)\, R\sin x)$

$R\cos(x + h) \approx R\cos x - (2/\Delta)\,(R\sin x + (1/\Delta)\, R\cos x)$

where $\Delta = 2R/h$.

[63] *Tantrasaṅgraha*, 2.10-14.

Accurate Determination of Sines

Given an arc $s = Rx$, divide it into n equal parts and let the *piṇḍa-jyās* B_j, and *śaras* $S_{j-1/2}$, with $j = 0, 1…n$, be given by

$$B_j = R\sin(jx/n)$$

$$S_{j-1/2} = R\text{vers}\,[(j-1/2)x/n]$$

If α be the *samasta-jyā* (total chord) of the arc s/n, then

$$(B_{j+1} - B_j) - (B_j - B_{j-1}) = (\alpha/R)(S_{j-1/2} - S_{j+1/2}) = -(\alpha/R)^2 B_j$$

for $j = 1,2,…n$. Hence

$$S_{n-1/2} - S_{1/2} = (\alpha/R)(B_1 + B_2 + …. + B_{n-1})$$

$$B_n - n B_1 = -(\alpha/R)^2[B_1 + (B_1 + B_2) + … + (B_1 + B_2 + ….+ B_{n-1})]$$

$$= -(\alpha/R)(S_{1/2} + S_{3/2} + …. + S_{n-1/2} - nS_{1/2})$$

If B and S are the *jyā* and *śara* of the arc s, in the limit of very large n, we have as a first approximation

$$B_n \approx B,\ B_j \approx js/n,\ S_{n-1/2} \approx S,\ S_{1/2} \approx 0 \text{ and } \alpha \approx s/n.$$

Hence

$$S \approx (1/R)\,(s/n)^2\,(1 + 2 + ….+ n-1) \approx s^2/2R$$

$$B \approx n(s/n) - (1/R)^2(s/n)^3[\,1 + (1 + 2) + …+(1 + 2+…+ n-1)] \approx s - s^3/6R^2$$

Iterating these results we get successive approximations, leading to the following series given by the rule[64] *Nihatya cāpavargeṇa…*:

$$R\sin(s/R) = B = R\,[(s/R) - (s/R)^3/3! + (s/R)^5/5! - …]$$

$$R - R\cos(s/R) = S = R\,[(s/R)^2 - (s/R)^4/4! + (s/R)^6/6! - …]$$

While carrying successive approximations, the following result for *vāra-saṅkalitas* (repeated summations) is used:

$$\sum_{j=1}^{n} j(j + 1)...(j + k - 1)/k! = n(n + 1)…(n + k)/(k + 1)! \approx n^{k+1}/(k + 1)!$$

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[64] *Yuktidīpikā*, cited earlier, p.118.

The above series for *jyā* and *śara* can be employed to calculate them, without using the tabular values, by using the sequence of numerical parameters given by the formulae,[65] *vidvān* etc and *stena* etc. For example, if a quadrant of a circle is assigned the measure q = 5400', then for a given arc s, the corresponding *jyā* B is given accurately to the third minute by: [66]

$$B \approx s - (s/q)^3(u_3 - (s/q)^2(u_5 - (s/q)^2(u_7 - (s/q)^2(u_9 - (s/q)^2 u_{11}))))$$

where

u_3 = 2220'39''40''', u_5 = 273'57''47''', u_7 = 16'05''41''', u_9 = 33''06'''
and u_{11} = 44'''

Accurate sine values can be used to find an accurate estimate of the circumference given a certain value of the diameter.

Series for the square of sine given by the rule[67] *Nihatya cāpavargeṇa*…

$$\mathrm{Sin}^2\, x = x^2 - x^4/(2^2 - 2/2) + x^6/(2^2-2/2)(3^2-3/2) - \ldots$$

These squares can also be directly computed using the formulae[68] *Śaurirjayati*...

Computation of sines without using the radius

Two proofs of the *jīve-paraspara-nyāya*

$\mathrm{Sin}\,(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\mathrm{Cos}\,(x \pm y) = \cos x \cos y - (\pm) \sin x \sin y$

Later, another proof is given using the formulae for the product of diagonals in a cyclic quadrilateral. It is noted that the *jīve-paraspara-nyāya* can be used to compute tabular sines.

Cyclic Quadrilateral

First a derivation of the formulae for the area of a triangle, its altitude and *ābādhās*, the two intercepts of the base formed by the altitude from the vertex to the base.

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65 Attributed to Mādhava by Nīlakaṇṭha in his *Āryabhaṭīyabhāṣya*, *Gaṇitapāda*, cited earlier, p.151; see also, *Yuktidīpikā*, cited earlier, p.117-118.

66 Mādhava has also given the tabulated sine values (for arcs in multiples of 225') accurately to the third minute in the rule *Śreṣṭhaṁ nāma variṣṭhānāṁ*… (cited by Nīlakaṇṭha in his *Āryabhaṭīyabhāṣya*, *Gaṇitapāda*, cited earlier, p.73-74).

67 *Yuktidīpikā*, cited earlier, p. 119.

68 *Yuktidīpikā*, cited earlier, p. 119-120.

A derivation of the following formulae for the product and difference in the squares of two chords, in terms of the chords associated with the sum and difference of the corresponding arcs:

$\sin^2 x - \sin^2 y = \sin(x + y)\sin(x - y)$

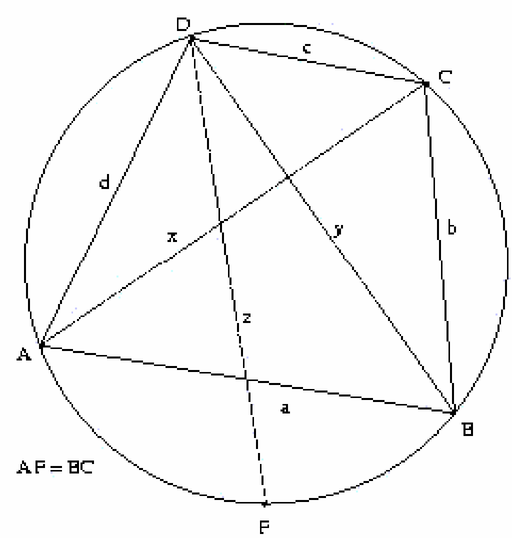
$\sin x \sin y = \sin^2[(x + y)/2] - \sin^2[(x - y)/2]$

Later, it is noted that these formulae can also be used to calculate tabular sines without using the radius.

Diagonals of a cyclic quadrilateral

If a, b, c, d are the sides of a cyclic quadrilateral and x, y, z are the three diagonals, then the above result on the product and difference of the squares of two chords is used to prove:

$$a.b + c.d = y.z$$
$$a.d + b.c = z.x$$
$$a.c + b.d = x.y$$



Hence

$$x = [(ac + bd)(ad + bc)/(ab + cd)]^{1/2}$$
$$y = [(ab + cd)(ac + bd)/(ad + bc)]^{1/2}$$
$$z = [(ab + cd)(ad + bc)/(ac + bd)]^{1/2}$$

It is noted that only three diagonals are possible.

Area of a cyclic quadrilateral

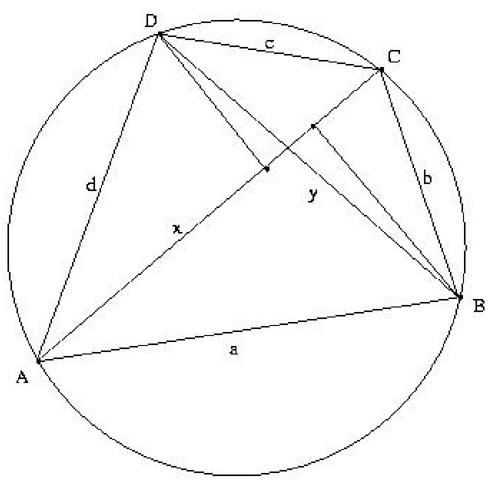
In the case of a triangle, by making use of the *jīve-paraspara-nyāya*, it is shown later that

Altitude = Product of the two adjacent sides/Circum-diameter

Based on this, it is shown that

Area of a cyclic quadrilateral = $(x.y.z) / 4R$

Area of a cyclic quadrilateral in terms of the sides, without using the diagonals or the circum-radius (*Līlāvatī* 169)



Draw altitudes from two corners to the opposite diagonal. Then

Area = (Sum of altitudes) (Diagonal)/2

$(\text{Sum of altitudes})^2 = (\text{Other diagonal})^2 - (\text{Distance between feet of the altitudes})^2$

$(\text{Area})^2 = [x^2 - \{(a^2 + c^2) - (b^2 + d^2)\}^2/4y^2](y^2/4)$

$= (s - a)(s - b)(s - c)(s - d)$

where, $s = (a + b + c + d)/2$ is the semi-perimeter of the quadrilateral[69].

69 The two results derived above for the area of a cyclic quadrilateral lead to the following formula for its circum-radius in terms of the sides (Parameśvara in his *Vivaraṇa* on *Līlāvatī*; also cited in *Kriyākramakarī*, cited earlier, p.363)

$$R = (1/4)[(ab + cd)(ac + bd)(ad + bc)/(s - a)(s - b)(s - c)(s - d)]^{1/2}$$

Similar proof is given of the formula for the area of a triangle

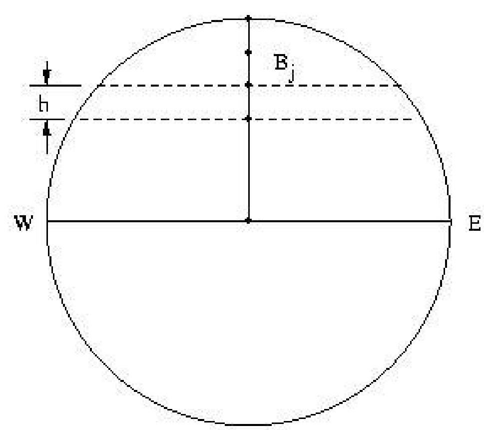
$$\text{Area of a triangle} = [s(s-a)(s-b)(s-c)]^{1/2}$$

Calculation of the of the *śaras* when two circles intersect (as given in *Āryabhaṭīya*, *Gaṇitapāda* 18)

Calculation of the two shadows from the same gnomon at different times (as given in *Līlāvatī* 232.)

Surface area of a sphere

Draw large number of circles at equal distances parallel to the East-West great circle. The radii of these circles are the sines B_j of the arcs of the great circle NWS. If h is the perpendicular distance between these circles



$$\text{Area} \approx 2\sum_{j=1}^{n} 2\pi B_j\, h$$

If Δ_j are sine differences, and α is the chord of each arc bit, then

$$B_j = (\Delta_j - \Delta_{j+1})(R/\alpha)^2$$

Since when n is very large, $\Delta_n \approx 0$, $\Delta_1 \approx \alpha \approx h$,

$$\text{Area} = 4\pi R^2$$

Volume of a sphere

First, a proof of the formula for the area of a circle.

Area = (½) Circumference x Radius

Divide the sphere into various slices parallel to the East-West circle as before. Then

$$\text{Volume} \approx \sum_{j} \pi B_j^2 h = \sum_{j} \pi h[(2R)^2/2 - j^2 h^2]$$

Since

$$h = 2R/n$$

and

$$B_j^2 = [(2R)^2/2 - j^2 h^2],$$

we get

$$\text{Volume} \approx 4\pi R^3 - \pi (2R/n)^3 \sum_{j=1}^{n} j^2$$

$$= (4/3)\pi R^3$$